Technical University of Cluj-Napoca

Basic of Electrotechnics

Recommended textbooks

- Radu Ciupa, Vasile Topa, “The Theory of Electric Circuits”, *Casa Cartii de Stiinta Printing House*, Cluj-Napoca, 1998;
- Vasile Topa, Radu Ciupa “The Theory of Electromagnetic Field”,
- PowerPoint presentations. Available on the web page http://www.et.utcluj.ro/Cursuri_V_Topa.htm

The Theory of Electric Circuits

Contents

1. Transient Regime of Linear Circuits
2. Transmission Lines

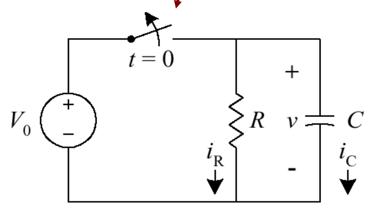
Transient Regime of Linear Circuits

Transient Regime of Linear Circuits

Definition

Whenever a circuit is switched from one condition to another, either by a change in the applied source or a change in the circuit elements – there is a transitional period during which the branch currents and element voltages change from their former values to new ones.

This period is called the **transient time.**



Transient Regime of Linear Circuits

Important Concepts

- The differential equation (the 1^{st}, 2^{nd}, …n^{th} order).
- Forced (particular) and natural (complementary) solutions.
- The time constant.
- Transient and steady state waveforms.

Transient Regime of Linear Circuits

Remarks

After the transient time (transient voltages and currents have settled down) the circuit is said to be in the steady state.

The time varying voltage and current in a circuit can be described by a linear differential equation:

$$\frac{d^n v(t)}{dt^n} + a_{n-1}\frac{d^{n-1}v(t)}{dt^{n-1}} + \ldots + a_0 v(t) = f(t)$$

n th order is given by the number of energy storage elements (capacitors and inductors)

Transient Regime of Linear Circuits

Particular Solution (Steady State Solution)

The particular solution – called also as forced solution *or* steady state solution – $v_p(t)$ is typically a weighted sum of $f(t)$ and its first n derivatives.

If the right term term $f(t)$ is:

- constant, then $v_p(t)$ is constant.
- sinusoidal, then $v_p(t)$ is sinusoidal.

Transient Regime of Linear Circuits

Complementary Solution (Transient Solution)

- The complementary solution – called also as transient solution *or* natural solution – is the solution to the homogeneous equation:

$$\frac{d^n v(t)}{dt^n} + a_{n-1}\frac{d^{n-1} v(t)}{dt^{n-1}} + ... + a_0 v(t) = 0$$

- The final complementary solution has the form:

$$v_t(t) = \sum_{i=1}^{n} K_i e^{s_i t}$$

Transient Regime of Linear Circuits

Where:

- s_1 through s_n are the roots of the characteristic equation:

$$s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 = 0$$

- if s_i is a real root, it corresponds to an exponential term:

$$K_i e^{s_i t}$$

- if s_i is a complex root, there is another complex root that is its complex conjugate, and together they correspond to an exponentially decaying sinusoidal term:

$$e^{-\sigma_i t}\left(A_i \cos\omega_d t + B_i \sin\omega_d t\right)$$

Final solution

$$\frac{d^n v(t)}{dt^n} + a_{n-1}\frac{d^{n-1} v(t)}{dt^{n-1}} + \ldots + a_0 v(t) = f(t)$$

unknowns

$$v = v_p(t) + v_t(t) = v_p(t) + \sum_{i=1}^{n} K_i e^{s_i t}$$

Continuity Conditions

Is it possible for the capacitor voltage or for the inductor current simply to jump up to their final values immediately ?

$$u_L = L\frac{di}{dt}$$

If, the current in an inductor is not continuous the voltage on the inductor is infinite.

The current in an inductor, immediately after a step in a source waveform must be the same as the current just before the step.

$$i_L(0_-) = i_L(0_+) = i_L(0)$$

Transient Regime of Linear Circuits

$$i_C = C\frac{du_c}{dt}$$

If, the voltage on a capacitor is not continuous the current across is infinite.

The voltage on a capacitor, immediately after a step in a source waveform must be the same as the voltage just before the step.

$$u_c(0_-) = u_c(0_+) = u_c(0)$$

The continuity conditions tell us that capacitor voltages and inductor currents cannot be discontinuous.

Transient Regime of Linear Circuits

Therefore, for a network with:

- zero initial conditions at t = 0, we need simply to replace a capacitor by a voltage source of zero voltage (this is simply a short circuit) ***or*** an inductor by a current source of zero output current (this is simply an open circuit), and solve for the initial values of any network variable in which we are interested.
- if some initial condition happens to be present on any of these elements, the replacement source will simply have the appropriate value of the initial capacitor voltages or inductor currents.

Transient Regime of Linear Circuits

$$v = v_p(t) + v_t(t) = v_p(t) + \sum_{i=1}^{n} K_i e^{s_i t}$$

$$i_L(0_-) = i_L(0_+) = i_L(0)$$

$$u_c(0_-) = u_c(0_+) = u_c(0)$$

For each:

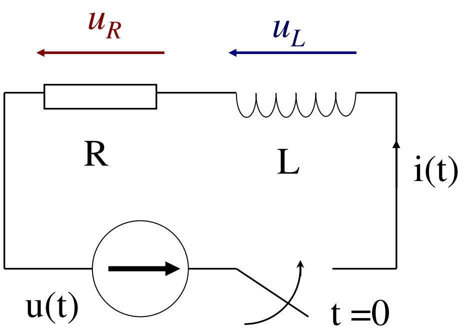
- Capacitor
- Inductor

First Order Linear Circuits

First Order Linear Circuits

- any circuit with a single energy storage element (inductors or capacitors), an arbitrary number of sources, and an arbitrary number of resistors is a circuit of order 1.
- types of 1st order circuits:
  - R, L circuits;
  - R, C circuits.
- any voltage or current in such a circuit is the solution to a 1st order differential equation.

Transient of the RL Circuit



$$u_R + u_L = u(t)$$

$$Ri + L\frac{di}{dt} = u(t)$$

$$i(t) = i_t(t) + i_p(t)$$

$$Ri_t + L\frac{di_t}{dt} = 0$$

Transient of the RL Circuit

Complementary Solution

- The characteristic equation:

$$s \cdot L + R = 0 \quad \Rightarrow \quad s = -\frac{R}{L}$$

$$i_t(t) = K \cdot e^{s \cdot t}, \quad t \geq 0$$

- How do I choose the value of *K*?
- The initial conditions determine the value of *K* (initial value of the inductor's current).

Transient of the RL Circuit

$$i(0) = i_p(0) + i_t(0) \quad or \quad i_0 = i_{P0} + K$$

$$Thus: K = i_0 - i_{p0}$$

$$i(t) = i_p(t) + (i_0 - i_{p0}) \cdot e^{-\frac{R}{L}t} = i_p(t) + (i_0 - i_{p0}) \cdot e^{-\frac{t}{\tau}}$$

Where:

$i_p(t)$	represents the solution of the non-homogenous equation (similar form to the excitation) - it is called the particular (or steady-state) solution
i_0	represents the value of the current immediately before the switching operation $i_0 = i(0_-)$. Obtained from the continuity conditions.
i_{p0}	represents the steady state-value of the current at t = 0
τ	the quantity *L/R* must have the dimensions of time (constant time).

Transient of the RL Circuit

The quantity L/R is called time constant of a R,L circuit

$$[\tau] = \frac{L}{R}$$

This is can be verified by direct dimensional analysis:

$$\left[\frac{L}{R}\right] = \frac{[\omega L]}{[R]} \cdot \frac{1}{[\omega]} = \frac{\Omega}{\Omega} \cdot \frac{1}{s^{-1}} = s$$

Transient of the RL Circuit (constant excitation)

Response to sources with constant excitation

a) The switch is closed

$$L\frac{di}{dt}+R\cdot i=E \qquad \text{U(t)} = E = cst.$$

- the steady-state current is: $i_p = E/R$
- the steady-state current at t = 0 is: $i_{p0} = i_p = E/R$
- the initial current in the circuit at t < 0 is: $i_0 = i(0-) = 0.$

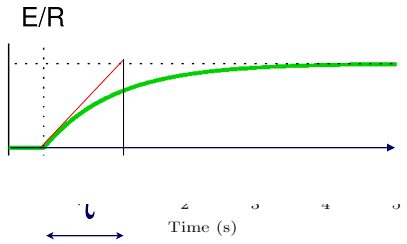
$$i(t)=i_p(t)+(i_0 - i_{p0})\cdot e^{-\frac{R}{L}t}$$

$$i(t)=\frac{E}{R}+(0-\frac{E}{R})\cdot e^{-\frac{R}{L}t}=\frac{E}{R}\cdot(1-e^{-\frac{R}{L}t})$$

Transient of the RL Circuit (constant excitation)

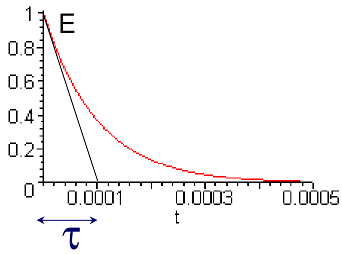
What does *i (t)* look like?

$$i(t)=\frac{E}{R}\cdot\left(1-e^{-\frac{R}{L}t}\right)$$



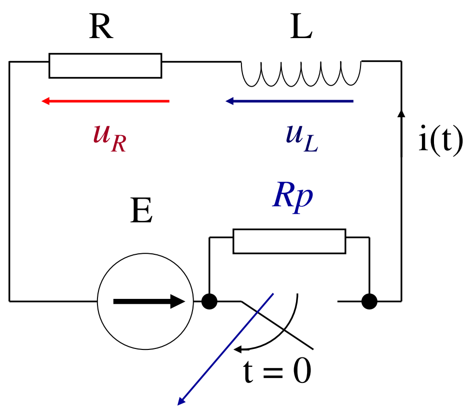
The voltage across the inductance L is:

$$u_L = L\cdot\frac{di}{dt}=E\cdot e^{-\frac{R}{L}t}$$



Transient of the RL Circuit (constant excitation)

b) The switch is opened



$$i(t)=i_p(t)+(i_0-i_{p0})\cdot e^{-\frac{1}{\tau}t}$$

$$i_p(t)=i_{p0}(t)=\frac{E}{R+R_p}$$

$$i_0=\frac{E}{R}$$

Rp = resistance between the switch contacts (non-linear)

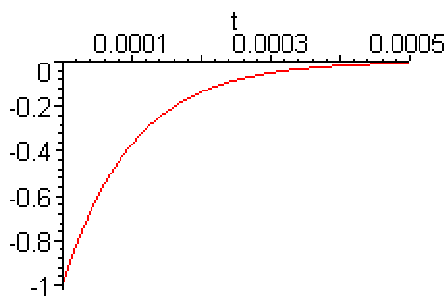
$$i(t)=\frac{E}{R+R_p}(1-e^{-\frac{1}{\tau}t})+\frac{E}{R}$$

Transient of the RL Circuit (constant excitation)

$$i(t)=\frac{E}{R+R_p}\cdot(1-e^{-\frac{1}{\tau}t})+\frac{E}{R}$$

The voltage across the inductance is:

$$u_L=L\cdot\frac{di}{dt}=-\frac{R_p}{R}\cdot E\cdot e^{-\frac{1}{\tau}t}$$



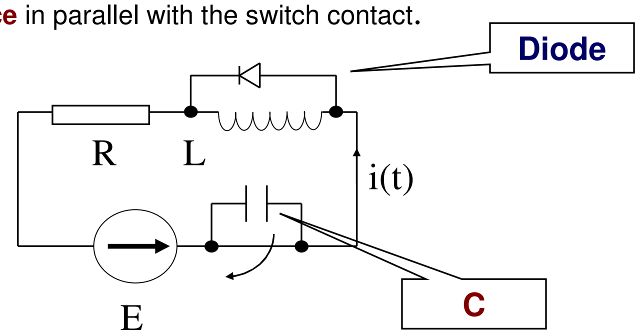
Transient of the RL Circuit (constant excitation)

Remarks

- if **E = 100 V** and **Rp/R** high (i.e. **10**) the voltage on the inductance and at the switch contacts is **1.000 V.**

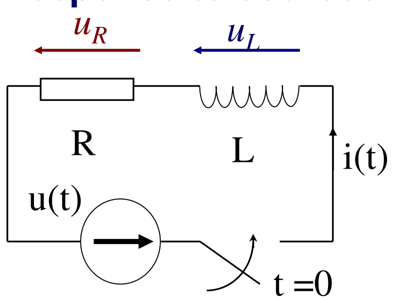
The switch contacts can be protected by:

- connecting a **diode** in parallel with the inductance, ***or***
- a **capacitance** in parallel with the switch contact.



Transient of the RL Circuit (sinusoidal - excitation)

Response to sources with sinusoidal excitation



$$u(t) = E \cdot \sqrt{2} \cdot \sin(\omega t + \gamma_e)$$

$$L\frac{di}{dt} + Ri = u(t)$$

$$i(t) = i_p(t) + (i_0 - i_{p0})e^{-\frac{R}{L}t}$$

$$i_p = \sqrt{2} \cdot I \cdot \sin(\omega t + \gamma_e - \varphi)$$

The steady state solution

$$\begin{cases} I = \dfrac{E}{\sqrt{R^2 + (\omega L)^2}} \\ \varphi = \tan^{-1}\dfrac{\omega L}{R} \end{cases}$$

Transient of the RL Circuit (sinusoidal - excitation)

The steady state solution at t = 0 $i_{p0} = \sqrt{2} \cdot I \cdot \sin(\gamma_e - \varphi)$

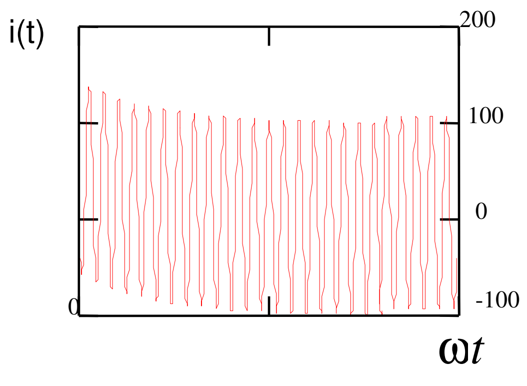
The initial value of the current t < 0 $i_0 = 0$

The total response for t > 0 is:

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega t + \gamma_e - \varphi) - \sqrt{2} \cdot I \cdot \sin(\gamma_e - \varphi) \cdot e^{-\frac{R}{L}t}$$

Transient of the RL Circuit (sinusoidal - excitation)

What does *i (t)* look like?



Transient of the RL Circuit (sinusoidal - excitation)

One can study two cases:

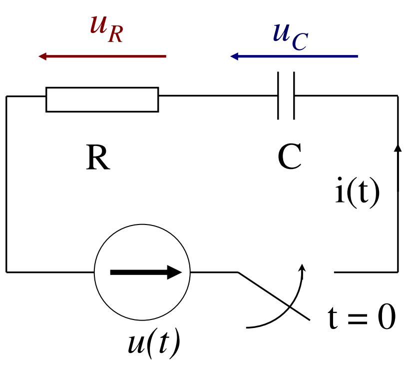
1) $\gamma - \varphi = 0$ ⇒ $i(t) = \sqrt{2} \cdot I \cdot \sin \omega t = i_p$

Thus the *steady-state regime appears immediately, without a transient response*.

2) $\gamma_e - \varphi = \frac{\pi}{2}$ ⇒ $i(t) = \sqrt{2} \cdot I \cdot (\cos \omega t - e^{\frac{-R}{L}t})$

Transient of the RC Circuit

Transient of the RC Circuit



$$u_R + u_C = u(t)$$

$$R \cdot i + \frac{1}{C} \cdot \int i dt = u(t)$$

But: $i = \frac{dq}{dt}$, $\int i\, dt = q$

$$R\frac{dq}{dt} + \frac{1}{C} q = u(t)$$

$$q(t) = q_p(t) + q_t(t)$$

Transient of the RC Circuit

Complementary Solution

The characteristic equation:

$$\frac{1}{C} + s \cdot R = 0 \qquad s = -\frac{1}{R \cdot C}$$

$$q_t(t) = \boldsymbol{K} \cdot e^{s \cdot t}, \quad t \geq 0$$

- How do I choose the value of ***K*** ?
- The initial conditions determine the value of ***K***.

Transient of the RC Circuit

$$q(0) = q_p(0) + q_t(0) \quad or\ q_0 = q_{p0} + K$$

$$Thus: K = q_0 - q_{p0}$$

Where: $$q(t) = q_p(t) + (q_0 - q_{p0}) \cdot e^{-\frac{t}{R \cdot C}}$$

- $q(t)$ is the **transient charge** of the capacitor for t > 0;
- $q_p(t)$ is the **permanent charge** (or **steady-state charge**) of the capacitor, as a form similar to the particular form of the excitation;
- q_{p0} represents the value of the **steady-state charge for t = 0**.
- q_0 represents the value of the charge immediately before the switching operation: $q_0 = q(0-)$.(Obtained from the **continuity conditions**).
- the quantity RC must have the dimensions of time.

Transient of the RC Circuit

$$u(0) = u_p(0) + u_L(0) \quad or\; u_0 = u_{P0} + K$$

$$Thus: K = u_0 - u_{p0}$$

$$u(t) = u_p(t) + \left(u_0 - u_{p0} \right) \cdot e^{-\frac{t}{R \cdot C}}$$

Where:

- *u(t)* is the **transient voltage** of the capacitor for t > 0;
- $u_p(t)$ is the **steady-state voltage** of the capacitor;
- u_{p0} is the value of the **steady-state voltage for t = 0.**
- u_0 is the value of the voltage immediately before the switching operation: $u_0 = u(0-)$. (Obtained from the **continuity conditions**).
- τ the quantity RC must have the dimensions of time.

Transient of the RC Circuit

The quantity RC is called time constant of a R,C circuit:

$$[\tau] = R \cdot C$$

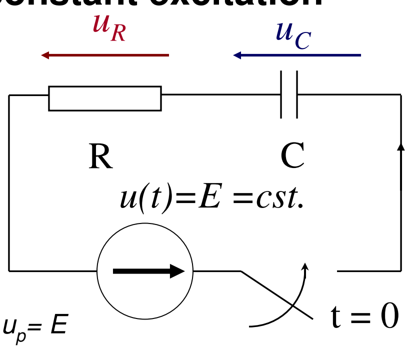
This is readily verified by direct dimensional analysis.

$$[\tau] = [RC] = \frac{[R]}{\left[\frac{1}{\omega C}\right][\omega]} = \frac{\Omega}{\Omega \cdot s^{-1}} = s$$

Transient of the RC Circuit (constant excitation)

Response to sources with constant excitation

a) The switch is <u>closed</u> (and the capacitor is with initial zero conditions)



$$u(t)=u_p(t)+\left(u_0 - u_{p0}\right)\cdot e^{-\frac{t}{R\cdot C}}$$

- the steady-state voltage is: $u_p = E$
- the steady-state voltage at t = 0 is: $u_{p0} = u_p = E$
- the initial voltage on the capacitor is: $u_0 = u_{c0}(0\text{-}) = 0$, for at t < 0.

$$u_C(t) = E + (0 - E)\cdot e^{-\frac{t}{R\cdot C}} = \frac{E}{R}\cdot\left(1 - e^{-\frac{t}{R\cdot C}}\right)$$

Transient of the RC Circuit (constant excitation)

The voltage u_C(t) on the capacitor is?

$$u_C(t) = E\cdot(1-e^{-\frac{t}{R\cdot C}})$$

The corresponding charge on the capacitor is:

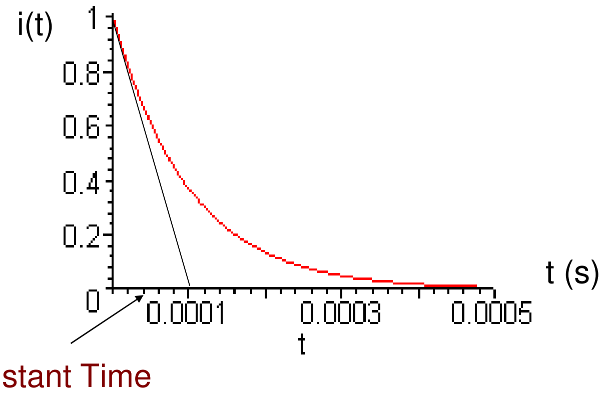
$$q_C = C\cdot u_c(t) = E\cdot C\cdot(1-e^{-\frac{t}{R\cdot C}})$$

The current is given by:

$$i=\frac{dq}{dt}=C\cdot\frac{du_C}{dt}=\frac{E}{R}\cdot e^{-\frac{t}{R\cdot C}}$$

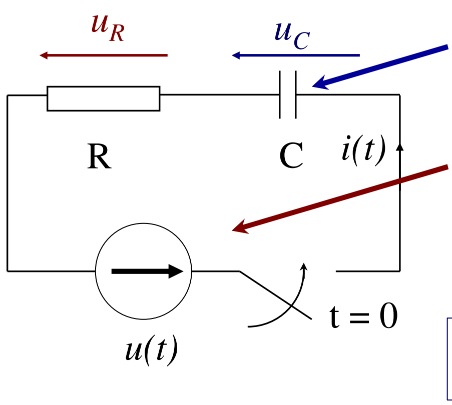
Transient of the RC Circuit (constant excitation)

What does i(t) look like?



Transient of the RC Circuit (sinusoidal excitation)

Response to sources with sinusoidal excitation



Let us consider u_{c0}= 0.

$$u(t) = E \cdot \sqrt{2} \cdot \sin(\omega t + \gamma_e)$$

$$u(t) = u_p(t) + \left(u_0 - u_{p0}\right) \cdot e^{-\frac{t}{R \cdot C}}$$

Transient of the RC Circuit (sinusoidal excitation)

The steady state solution of the current:

$$\begin{cases} Z = \sqrt{R^2 + \dfrac{1}{(\omega \cdot C)^2}} \\ \varphi = \tan^{-1} \dfrac{1}{R \cdot C \cdot \omega} \end{cases} \leftarrow \quad i_p = \sqrt{2} \cdot \frac{E}{Z} \cdot \sin(\omega t + \gamma_e - \varphi)$$

The steady state solution of the u_{CP}

$$u_{cp} = \frac{1}{C} \cdot \int i_p \cdot dt = \sqrt{2} \cdot \frac{E}{Z \cdot \omega \cdot C} \cdot \sin(\omega t + \gamma_e - \varphi - \frac{\pi}{2})$$

Transient of the RC Circuit (sinusoidal excitation)

The initial value of the voltage at t = 0

$$u_{cp0} = \sqrt{2} \cdot \frac{E}{Z \cdot \omega \cdot C} \cdot \sin(\gamma_e - \varphi - \frac{\pi}{2})$$

The total response is:

$$u_c = \sqrt{2} \frac{E}{Z \cdot \omega \cdot C} \left[\sin(\omega t + \gamma_e - \varphi - \frac{\pi}{2}) - \sin(\gamma_e - \varphi - \frac{\pi}{2}) e^{-\frac{t}{RC}} \right]$$

Transient of the RC Circuit (sinusoidal excitation)

One can study two cases:

1) $\gamma_e - \varphi = \frac{\pi}{2}$ → $u_C(t) = \sqrt{2}\cdot\frac{E}{Z\cdot\omega\cdot C}\cdot \sin\omega t = u_p(t)$

Thus the steady-state regime appears immediately, without a transient response.

2) $\gamma_e - \varphi = 0$ → $u_C(t) = \sqrt{2}\cdot\frac{E}{Z\cdot\omega\cdot C}\cdot\left(-\cos\omega t + e^{-\frac{t}{RC}}\right)$

If $t << RC$,
This is the, so-called, **stroke voltage.**

$$u_{c_{\max}} \cong 2\left(\sqrt{2}\frac{E}{Z\cdot\omega\cdot C}\right) = 2\cdot U_m$$

Constant time

Constant Time

The complementary solution for any 1st order circuit is:

$$v_c(t) = K\cdot e^{-t/\tau}$$

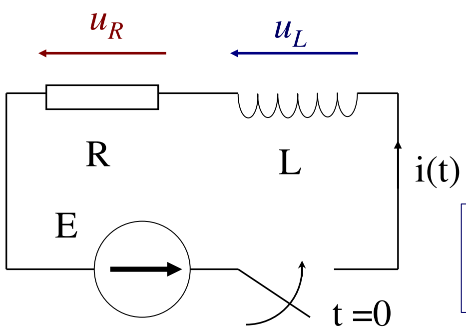
1) For R, L circuit, constant time $\tau = \frac{L}{R}$

2) For R, C circuit, constant time $\tau = R\cdot C$

Constant Time

Definition

The **time constant** of a given circuit is defined as the **time required** for any variable to decay to 36.8% of its initial value when the circuit is excited only by initial conditions. If we consider the circuit shown in the figure the solution for $u_L(t)$ is:



$$u_L(\tau) = L \cdot \frac{di}{dt} = E \cdot e^{-\frac{R}{L}\tau}$$

$$u_L\left(\frac{L}{R}\right) = E \cdot e^{-1} = 0.368 \cdot E$$

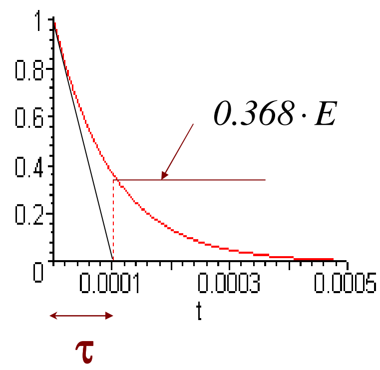
Constant Time

Interpretation of τ

1/ τ is the initial slope of an exponential with an initial value of E = 1 V.

$$u_L = L \cdot \frac{di}{dt} = E \cdot e^{-\frac{t}{\tau}}$$

$$\frac{d(u_L)}{dt} = \left(-\frac{1}{\tau}\right) \cdot E \cdot e^{-\frac{0}{\tau}} = -\frac{1}{\tau} \cdot E.$$



Constant Time

Interpretation of τ

τ represents the time required for *i* (t) to reach 63,2 % of its final value in an 1st order R, L circuit.

$$i(\tau)=\frac{E}{R}\cdot\left(1-e^{-\frac{\tau}{\tau}}\right)=\frac{E}{R}\cdot\left(\frac{e-1}{e}\right)=i_p\cdot\left(\frac{e-1}{e}\right)=i_p\cdot 0{,}632$$

$$i(2\cdot\tau)=i_p\cdot 0{,}865$$
$$i(3\cdot\tau)=i_p\cdot 0{,}950$$
$$i(4\cdot\tau)=i_p\cdot 0{,}982$$
$$i(5\cdot\tau)>i_p\cdot 0{,}99\cong steady\ \ state\ \ regim$$

Second Order Linear Circuits

Second Order Linear Circuits

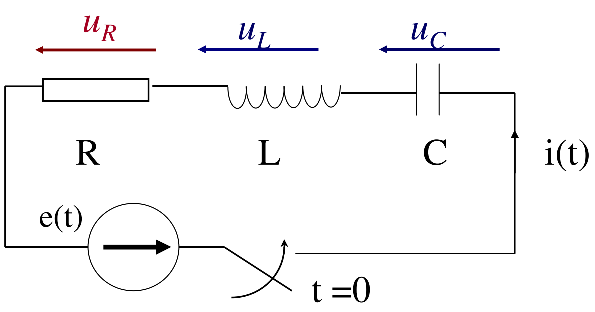
- any circuit with a single capacitor, a single inductor, an arbitrary number of sources, and an arbitrary number of resistors is a circuit of order 2.
- type of 2nd order circuit:
  - R, L, C circuits.
- any voltage or current in such a circuit is the solution to a 2nd order differential equation.

Second Order Linear Circuits

Important Concepts

- The differential equation of the 2nd order
- Forced and homogeneous solutions
- The natural frequency and the damping ratio
- Transient and steady state waveforms

Second Order Linear Circuits



Applying KVL to the RLC series circuit we obtain:

$$u_R + u_L + u_C = e$$

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt = e(t)$$

Second Order Linear Circuits

Because one may write:

$$i = \frac{dq}{dt}, \qquad \int i dt = q, \qquad \frac{di}{dt} = \frac{d^2 q}{dt^2}$$

$$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t)$$

e(t) = 0

The complete solution of the equation is given by:

$$q(t) = q_t(t) + q_p(t)$$

Second Order Linear Circuits

$$\frac{d^2 q}{dt^2} + \frac{R}{L}\cdot\frac{dq}{dt} + \frac{1}{L\cdot C}q = \frac{1}{L}e(t)$$

Most circuits with one capacitor and inductor are not as easy to analyze as the previous circuits. However, every voltage and current in such a circuit is the solution to a differential equation of the following form:

$$\frac{d^2 i(t)}{dt^2} + 2\zeta\omega_0\frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)$$

Second Order Linear Circuits

The damping ratio is: $\varsigma = \frac{R}{2} \cdot \sqrt{\frac{C}{L}}$

The natural frequency is: $\omega_0 = \frac{1}{\sqrt{L \cdot C}}$

$$i(t) = i_t(t) + i_p(t)$$

The transient solution (or homogenous solution) has the form:

$$i_t(t) = K \cdot e^{st}$$

- ***K*** is a constant determined by the initial conditions.
- ***s*** is the root of the homogenous equation (function of the coefficients of the differential equation): $s = s(R, L, C)$

Second Order Linear Circuits

Characteristic Equation

To find the transient solution, we need to solve the characteristic equation:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

- The characteristic equation has two roots, s_1 and s_2.
- The transient solution is:

$$i_t(t) = K_1 \cdot e^{s_1 t} + K_2 \cdot e^{s_2 t}$$

Second Order Linear Circuits

Damping Ratio and Natural Frequency

- **The damping ratio** is ζ. The damping ratio determines what type of solution we will get:
  - exponentially decreasing ($\zeta > 1$)
  - exponentially decreasing sinusoid ($\zeta < 1$)
- **The natural frequency** is ω_0
  - it determines how fast sinusoids wiggle.

Second Order Linear Circuits

Roots of the Characteristic Equation

The roots of the characteristic equation determine whether the transient solution wiggles.

$$s_1 = -\varsigma\omega_0 + \omega_0\sqrt{\varsigma^2 - 1} \qquad s_2 = -\varsigma\omega_0 - \omega_0\sqrt{\varsigma^2 - 1}$$

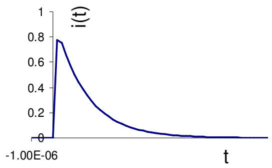
a) Real Unequal Roots ($\zeta > 1$)

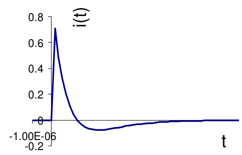
$$i_t(t) = K_1 \cdot e^{\left(-\varsigma\omega_0 + \omega_0\sqrt{\varsigma^2 - 1}\right)t} + K_2 \cdot e^{\left(-\varsigma\omega_0 - \omega_0\sqrt{\varsigma^2 - 1}\right)t}$$

Remarks:
This solution is over damped.
The constants K_1 , K_2 are computed from the initial conditions of *L* and *C*.

Second Order Linear Circuits

Over damped examples:





b) Real Equal Roots ($\zeta = 1$) ➡ $i_t(t) = (K_1 + K_2) \cdot e^{-\omega_0 t}$

Remarks:

This solution is critically damped.

c) Complex Roots ($\zeta < 1$).

Define the following constants:

$$\sigma = \varsigma \omega_0$$

$$\omega_d = \omega_0 \sqrt{1 - \varsigma^2}$$

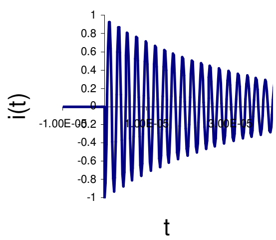
Second Order Linear Circuits

$$i_t(t) = e^{-\sigma t} \cdot (A_1 \cdot \cos \omega_d t + A_2 \cdot \sin \omega_d t)$$

Remarks:

- This solution is under damped.
- The constants A_1 and A_2 are computed from the initial conditions of the reactive elements L and C.

Under damped example:



Second Order Linear Circuits

Example

$$\omega_0 = 2\pi 455000$$

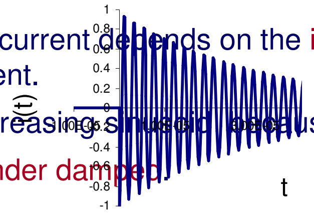
$$\zeta = 0.011$$

1. Is this system over damped, under damped, or critically damped?
2. What will the transient current look like?

a) The shape of the current depends on the initial capacitor voltage and inductor current.

b) Exponentially decreasing sinusoid, because $\zeta < 1$.

c) This solution is under damped.



Higher Order Linear Circuits

Higher Order Linear Circuits

- The text has a chapter on 1st order circuits and a chapter on 2nd order circuits.
- The text has no chapter on 3rd order circuits.
- Why ?

Higher Order Linear Circuits are Boring!

- The behavior of a higher order (3rd or higher order) circuit is not qualitatively different than that of a 1st or 2nd order.
- Particular solutions are similar, especially for constant and sinusoidal sources.

Higher Order Linear Circuits

Remarks:

1. The natural response is a sum of decaying exponentials and/or exponentially decaying sinusoids.
2. The responses of higher order circuits have the same sort of characteristics as 1st and 2nd order circuits
3. There are more terms in the solution.

Mathematical Justification

Any voltage or current in an *n*th order linear circuit is the solution to a differential equation of the following form:

$$\frac{d^n v(t)}{dt^n} + a_{n-1}\frac{d^{n-1}v(t)}{dt^{n-1}} + \ldots + a_0 v(t) = f(t)$$

Higher Order Linear Circuits

$$v(t) = v_p(t) + v_t(t)$$

1. The particular solution $v_p(t)$ is typically a weighted sum of $f(t)$ and its first *n* derivatives.

$$f(t) = \begin{cases} constant \Rightarrow v_p(t) = constant \\ sinusoidal \Rightarrow v_p(t) = sinusoidal \end{cases}$$

2. The transient solution is the solution of the equation:

$$\frac{d^n v(t)}{dt^n} + a_{n-1}\frac{d^{n-1}v(t)}{dt^{n-1}} + \ldots + a_0 v(t) = 0 \Rightarrow v_t(t) = \sum_{i=1}^{n} K_i e^{s_i t}$$

s_1 - s_n are the roots of the characteristic equation

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

K_i are the constants obtained from continuity conditions

Higher Order Linear Circuits

Time Waveforms

If s_i is a real root, it corresponds to an exponential term $K_i e^{s_i t}$

If s_i is a complex root, there is another complex root that is its complex conjugate, and together they correspond to an exponentially decaying sinusoidal term

$$e^{-\sigma_i t}\left(A_i \cos\omega_d t + B_i \sin\omega_d t\right)$$

Higher Order Linear Circuits

Example 1

A 3rd order linear circuit has the following characteristic equation:

$$s^3 + 6s^2 + 11s + 6 = 0$$

1. What terms would we expect in the transient solution?

Answer

The roots of the characteristic equation are: -1, -2, and -3

The transient solution is: $K_1 e^{-t} + K_2 e^{-2t} + K_3 e^{-3t}$

Initial conditions will determine the values of the constants K.

Higher Order Linear Circuits

Example 2

A 4th order linear circuit has the following characteristic equation:

$$s^4 + s^3 - 2s^2 + 2s + 4 = 0$$

1. What terms would we expect in the transient solution?

Answer

The roots of the characteristic equation are: -1, -2, (-1 + j), (-1 – j).

The transient solution is: $K_1 e^{-t} + K_2 e^{-2t} + e^{-t}(A_3 \cos t + B_3 \sin t)$

Initial conditions will determine the values of the constants K.

The Laplace Transform

The Laplace Transform

The advantages of the Laplace transform for the analysis of feedback systems are:

1. It includes the initial conditions.
2. The work involved in the solution is a simple algebra.
3. The work is systematized.
4. The use of a table of transforms reduces the effort required.
5. The discontinuous inputs can be treated.
6. The transient and steady-state components of the solution are obtained simultaneously.

The Laplace Transform

Definition

The Laplace transform is defined as: $\mathcal{L}[f(t)] = \int_0^\infty f(t)\cdot e^{-st}\cdot dt = F(s)$

where:

- *s is a complex quantity*
- *f(t) time domain function*
- *F(s) frequency domain function.*

Conditions that the integral converges to a finite value:

$\int_0^\infty |f(t)|\, e^{-\sigma_1 t}\, dt < \infty$

$f(t)$ to be continuous after $t = 0$.

$f(t) = 0$ for $t < 0$.

The Laplace Transform

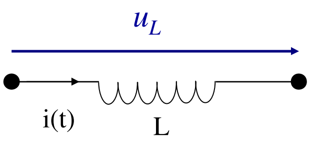
Important Laplace Transform

$f(t)$	$F(s)$
k	$\frac{k}{s}$
$e^{-at}\,(a>0)$	$\frac{1}{s+a}$
$\sin\ \omega t$	$\frac{\omega}{s^2+\omega^2}$
t	$\frac{1}{s^2}$
$e^{-at}\sin\ \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$

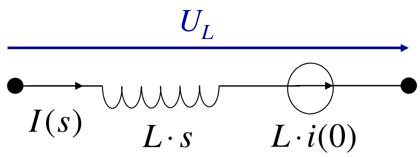
Basic Laplace Transform Operations

$f(t)$	$F(s)$
$f_1(t)+f_2(t)$	$F_1(s)+F_2(s)$
$k\cdot f(t)$	$k\cdot F(s)$
$\frac{df}{dt}$	$s\cdot F(s) - f(0-)$
$\int_{-\infty}^{t} f dt$	$\frac{1}{s}F(s)+\frac{f(0-)}{s}$

Network Analysis by Laplace Transform

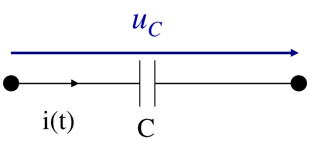


$\Rightarrow$

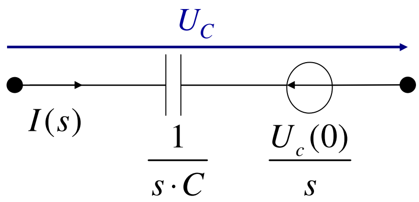


$$u_L(t) = \frac{L\cdot di(t)}{dt} \Rightarrow$$

$$U_L(s) = s\cdot L\cdot I(s) - L\cdot i(0)$$



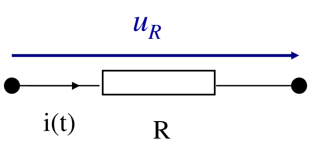
$\Rightarrow$



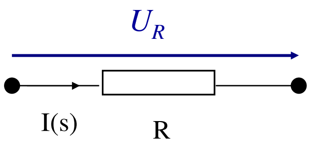
$$u_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(t) \Rightarrow$$

$$U_C(s) = \frac{1}{s\cdot C}\cdot I(s) + \frac{U_c(0)}{s}$$

Network Analysis by Laplace Transform



$\Rightarrow$



$$u_R(t) = R\cdot i(t) \Rightarrow$$

$$U_R(s) = R\cdot I(s)$$

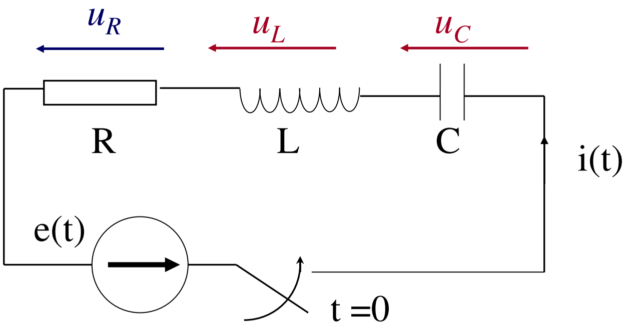
Generalized impedance

$$Z_R = R \qquad Z_L = s\cdot L \qquad Z_C = \frac{1}{s\cdot C}$$

Analogy

$$Z_R = R \qquad Z_L = j\cdot\omega\cdot L \qquad Z_C = \frac{1}{j\cdot\omega\cdot C}$$

The Laplace Transform

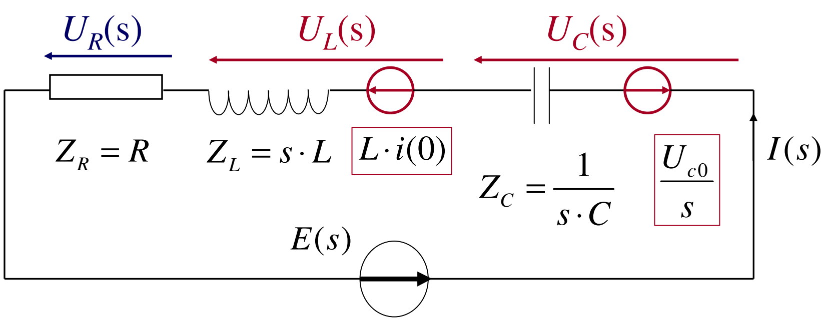


Applying KVL to the RLC series circuit we obtain:

$$u_R(t) + u_L(t) + u_C(t) = e(t)$$

$$R \cdot i + L \cdot \frac{di}{dt} + \frac{1}{C}\int i \cdot dt = e(t)$$

Network Analysis by Laplace Transform



$$U_R(s) + U_L(s) + U_C(s) = E(s)$$

$$R \cdot I(s) + s \cdot L \cdot I(s) - L \cdot i(0) + \frac{1}{s \cdot C} \cdot I(s) + \frac{U_{C0}}{s} = E(s)$$

$$\left(R + s \cdot L + \frac{1}{s \cdot C}\right) \cdot I(s) = E(s) + L \cdot i(0) - \frac{U_{C0}}{s}$$

The Laplace Transform

$$\left(R + s \cdot L + \frac{1}{s \cdot C}\right) \cdot I(s) = E(s) + L \cdot i(0) - \frac{U_{C0}}{s}$$

⬇

$$Z(s) \cdot I(s) = E(s) + L \cdot i(0) - \frac{U_{C0}}{s}$$

⬇

$$Z(s) \cdot I(s) = E(s)$$

If the **initial conditions are zero**:

$$i(0) = U_{C0} = 0$$

Network Analysis by Laplace Transform

Initial circuit in time domain
(at t < 0)

⬇

Computation of the initial conditions at t < 0
$i_L(0)$ *and* $U_{C0}(0)$ *at* $t < 0$

⬇

Circuit in complex domain t > 0
(Laplace transform)

⬇

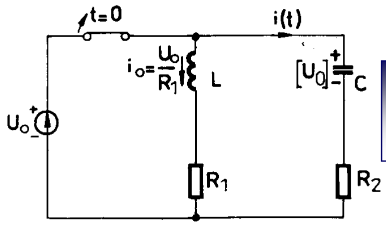
Analysis of the circuit in s domain
(any method can be applied)

⬇

Inverse Laplace transform ➡ Solutions of the circuit in time domain

Network Analysis by Laplace Transform

Example



Initial circuit in time domain (at t < 0)

$$i_L(0) = i_0 = \frac{U_0}{R_1}$$

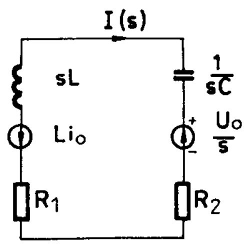
$$U_{C0} = U_C(0) = U_0$$

Computation of the initial conditions $i_L(0)$ *and* U_{C0} *at* $t < 0$

Circuit in complex domain t > 0 (Laplace transform)

Network Analysis by Laplace Transform

Circuit in complex domain t > 0 (Laplace transform)



Analysis of the circuit in s domain
(any method can be applied)

Possible methods:
Kirchhoff, loop, potential, Thevenin, Norton, etc.

$$I(s) = \frac{L \cdot i_0 - \frac{U_0}{s}}{R_1 + R_2 + s \cdot L + \frac{1}{s \cdot C}} \quad \Rightarrow \quad i(t) = ?$$

Network Analysis by Laplace Transform

$$I(s)=\frac{L\cdot i_0-\frac{U_0}{s}}{R_1+R_2+s\cdot L+\frac{1}{s\cdot C}}$$ ➡ $i(t)=?$

Inverse Laplace transform

Solutions of the circuit in time domain

Heaviside method

$$F(s)=\frac{P(s)}{Q(s)}=\frac{a_m s^m+a_{m-1}s^{m-1}+\cdots+a_1 s+a_0}{s^n+b_{n-1}s^{n-1}+\cdots+b_1 s+b_0}$$

$$F(s)=\frac{P(s)}{Q(s)}=\frac{P(s)}{(s-s_1)\cdot(s-s_2)\cdot(s-s_3)\cdots(s-s_k)\cdots(s-s_n)}$$

Network Analysis by Laplace Transform

Case 1: First-order real poles

$$f(t)=\frac{P(s_k)}{Q'(s_k)}\cdot e^{s_k t}$$

$$Q(s)=\prod_{k=1}^{n}(s-s_k)$$

Case 2: First-order real poles with one zero pole

$$Q(s)=s\cdot\prod_{k=1}^{n-1}(s-s_k)=s\cdot Q_1(s)$$

$$f(t)=\frac{P(0)}{Q_1(0)}+\frac{P(s_k)}{Q_1^{'}(s)}\cdot e^{s_k t}$$

Other cases: see the book

Network Analysis by Laplace Transform

$$I(s)=\frac{L\cdot i_0-\frac{U_0}{s}}{R_1+R_2+s\cdot L+\frac{1}{s\cdot C}} \Rightarrow i(t)=?$$

Suppose: $$I(s)=\frac{s+2}{(s+1)\cdot(s+4)}$$

$$P(s)=s+2$$
$$Q(s)=(s+1)(s+4)$$
$$Q'(s)=2s+5$$
$$s_1=-2$$
$$s_2=-4$$

$$i(t)=\frac{(-1+2)}{(2\cdot s+5)_{s=-1}}e^{-t}+\frac{(-4+2)}{(2\cdot s+5)_{s=-4}}e^{-4t}=\frac{1}{3}e^{-t}+\frac{2}{3}e^{-4t}$$