Transmission Lines

Topics of the course

- Limitation of the circuit theory
- Definition & types of transmission lines (TL)
- Primary TL parameters
- The distributed-element model of TL
- The first order equations (and second) of TL
- Derivation of power equation
- Steady-state TL equations.
- Equivalent TL equations
- Semi-infinite TL
- Heaviside TL conditions

Limitations of the circuit theory

At low frequencies, **circuit theory** is generally adequate to explain the behavior of collections of electronic components interconnected by wires. Such circuits are modeled using: **lumped circuits.**

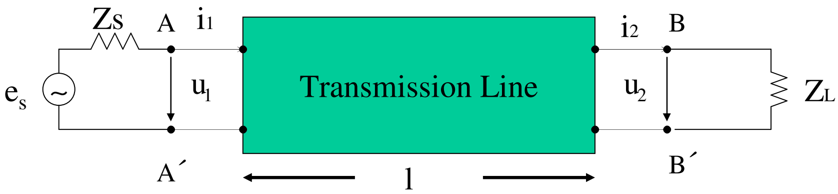
As frequency increases, the circuit approximation becomes progressively **less good**, for a number of reasons:

- the energy stored in **reactive components** is held in the space around the components, and different components can have "fields" which overlap;
- **wires** are also reactive components which store energy;
- the division of the circuit into separate reactive components interconnected by non-reactive "wires" is only an approximation.

Lumped circuits

- Assumes that the entire circuit is at a single point (lumped).
- This means that circuit component dimensions are unimportant.
- Voltage & current do not vary across the component.
- Voltage applied at one point, the rest of the circuit reacts instantly.
- We don’t have to consider travel time of the signal across components.
- Typical lumped elements are resistors, capacitors, inductors.

The Lumped-Element Model



- **Lumped-element** model for the entire transmission as seen from A-A´ and B-B´:

$$i_1 = i_2 \qquad \text{(KCL)}$$

$$u_1 = u_2 \qquad \text{(KVL)}$$

Valid only if

l (length of the line) << λ (wavelength of the signal)

Light speed

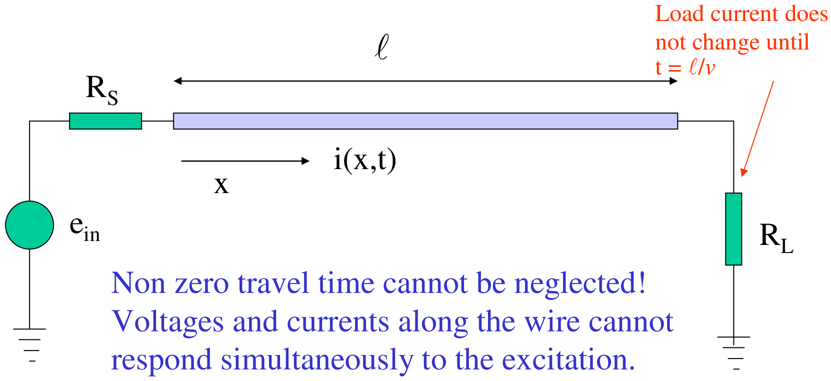
But ……the speed of light is finite and electromagnetic signals (**in free space**) travel at the light speed.

$$c_0 = 3\times10^8 \text{ m/sec}$$

In practical transmission systems the actual speed of the signal is determined by **electrical and magnetic properties** of the surrounding media and the **geometry** of the conductors.

So what does a distributed circuit look like?

A Long Wire!



Non zero travel time cannot be neglected!
Voltages and currents along the wire cannot respond simultaneously to the excitation.

When do we use lumped/distributed analysis?

For steady state sinusoidal applications we can look at the one way propagation delay time t_d.

Say we have an excitation $e_{in} = E \cdot \sqrt{2} \cdot sin(\omega t)$

Then the voltage at a distance of ℓ is delayed by $t_d = \ell / v$

This voltage is:

$$u_\ell(t) = E \cdot \sqrt{2} \cdot sin[\omega(t - t_d)]$$
$$= E \cdot \sqrt{2} \cdot sin\left[\omega t - 2\pi \frac{t_d}{T}\right]$$

T = 1/f = period

The delay is dependent with the frequencies !!

Two options

1. **t_d<<T**
   Then voltage at ℓ is almost the same as the input voltage.
   - treat the line as a **lumped system**.
2. **t_d comparable to T**
   In this case the voltage down the line can be quite different e.g. at $t_d = 0.5T$, $u_\ell = -E_{in}$ – **distributed system**.

Rules:
Use lumped analysis when:
- $t_d<0.1T$, or more stringently when $t_d<0.01T$ ***or equivalent***
- the wavelength of the electromagnetic signal is comparable with the geometric length of the line:

$$\lambda = c \cdot T = \frac{c}{f} = \frac{3\times10^8}{f} [\,m\,]$$

Maximum lengths for lumped applications

Application	Frequency	Wavelength $\lambda = \frac{3\times10^8}{f}$	Max length (based on t_d = 0.01T)
Power Transmission	**50 Hz**	**6.000 km**	**60 km**
Telephone	**1 kHz**	**300 km**	**3 km**
TV	**150 Mhz**	**2 m**	**2 cm**
Radar/Microwave	**10 GHz**	**3 cm**	**0.3 mm**
Visible light	**5×1014 Hz**	**600 nm**	**6 nm**

Introduction

The term transmission line is usually reserved for structures that are at least a significant fraction of a wavelength in length and have uniform electromagnetic properties along their length.

Examples are as follows:

- **coaxial lines**: flexible, semi flexible or rigid with solid insulation,perforated insulation or air-spaced;
- **parallel pair of conductors**: parallel wires (air-spaced, insulated); parallel tracks on an insulating substrate;
- **strip-line**: microstrip (in microwave integrated circuits): conducting strips insulated from a conducting ground plane;
- **waveguides**: hollow metal pipes (commonly of rectangular cross-section).

Main remarks about TL

- **Definition:**

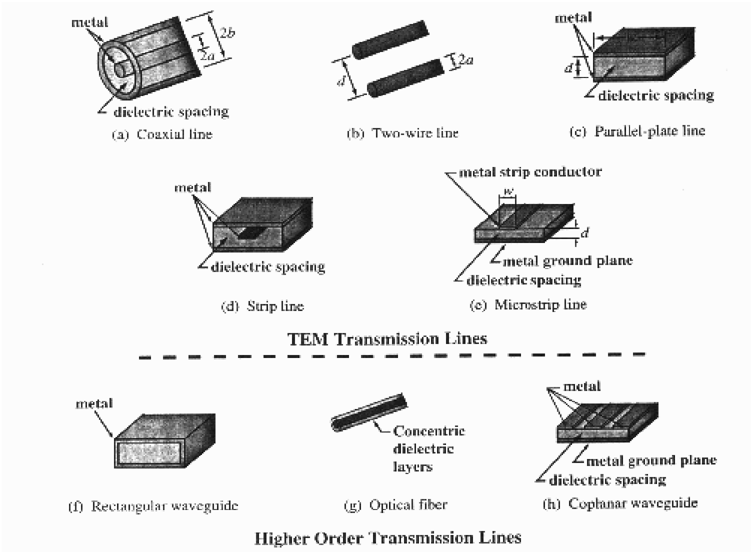
  Structures used to transmit energy or signal in the form of guided-wave electromagnetic fields from one place to another.
- **Modeling:**

  Distributed-element model

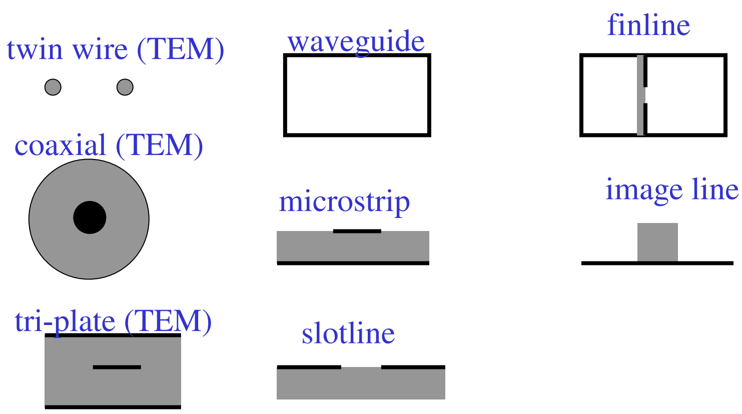
  Electromagnetic model (Maxwell equations)
- **Key Concepts:**

  Waves and properties

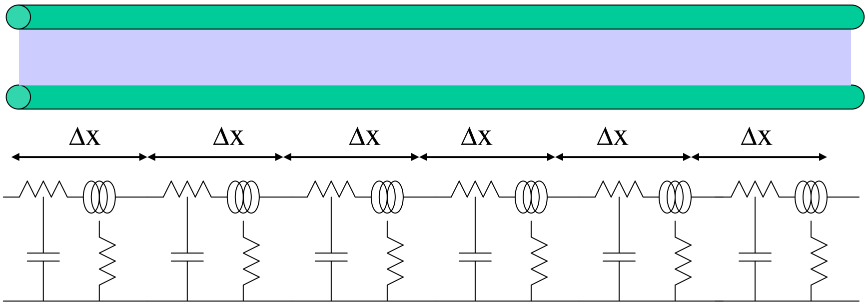
Different Types of Transmission Lines



Types of Transmission lines (cont)



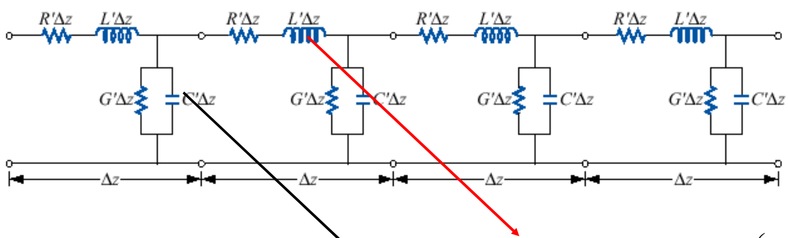
The Distributed-Element TL Model



1. Divide the entire transmission line into segments with length $\Delta x \ll \lambda$
2. For each segment, postulate the lumped elements related to the **series resistance**, the **parallel capacitance**, the **series inductance**, and the **parallel conductance**.

Distributed element-model

Inductance and Capacitance are the most important parameters.



The current in the line sets up a field and induces a voltage $\left(L_0 \frac{di}{dt}\right)$
The capacitance exists between the lines.
Both L_0 & C_0 are distributed along the line.
Both are not perfect, some attenuation (R_0 & G_0) always exists.

The primary TL parameters

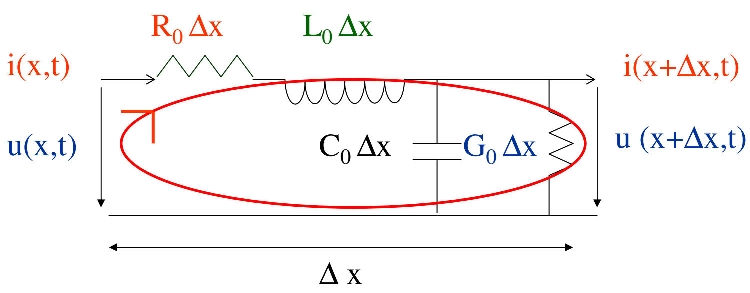
Any (two-wire) transmission line has four fundamental electrical parameters. They are also called primary line parameters (constants):

- R_0, the series resistance: $R_0 = \lim_{\Delta x \to 0} \frac{\Delta R}{\Delta x}$
- L_0, the series inductance: $L_0 = \lim_{\Delta x \to 0} \frac{\Delta L}{\Delta x}$
- C_0, the shunt capacitance: $C_0 = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$
- G_0, the shunt conductance (or leakage): $G_0 = \lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x}$

In general, these lines parameters are referred to a loop or pair of wires of fixed length or per unit length of line.

Transmission line equations

Consider one segment at position x along the line:



R_0 : the resistance of the conductors per unit length in Ω/m;
L_0 : the inductance of the conductors per unit length in H/m;
G_0 : the conductance of the insulation medium per unit length in 1/Ωm;
C_0 : the capacitance of the conductors per unit length in F/m.

Next Step: To establish relations among the currents and the voltages at x and x + Δx

By application of KVL:

$$-u(x,t)+u(x+\Delta x,t)+R_0\cdot\Delta x\cdot i+L_0\cdot\Delta x\cdot\frac{\partial}{\partial t}i=0$$

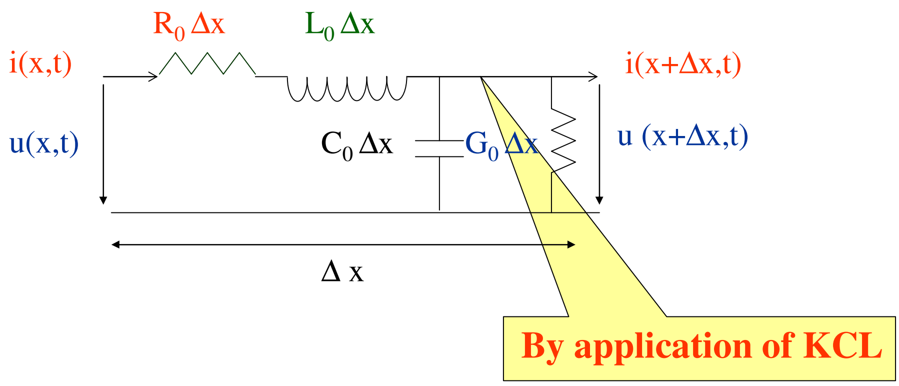
but:

$$u(x+\Delta x,t)=u(x,t)+\frac{\partial u}{\partial x}\cdot\Delta x$$

Divided by Δx and let $\Delta x \to 0$

$$-\frac{\partial u}{\partial x}=R_0\cdot i+L_0\cdot\frac{\partial i}{\partial t}$$

First order telegrapher's equations



$$-i(x,t)+i(x+\Delta x,t)+G_0\cdot\Delta x\cdot u+C_0\cdot\Delta x\cdot\frac{\partial}{\partial t}u=0$$

$$i(x+\Delta x,t)=i(x,t)+\frac{\partial i}{\partial x}\cdot\Delta x$$

$$-\frac{\partial i}{\partial x}=G_0\cdot u+C_0\cdot\frac{\partial u}{\partial t}$$

First order telegrapher's equations

Comments on the TL equations

$$-\frac{\partial u}{\partial x} = R_0 \cdot i + L_0 \cdot \frac{\partial i}{\partial t} \qquad -\frac{\partial i}{\partial x} = G_0 \cdot u + C_0 \cdot \frac{\partial u}{\partial t}$$

- both equations, within the level of approximations, are partial differential equations (PDE) that governing the voltage and the current along the transmission lines.
- the line primary parameters R_0, L_0, G_0, C_0 are related to the physical properties of the transmission line and may be functions of position x.
- give the line parameters, solutions of the equations describe the voltage and the current along the transmission lines

Second order telegrapher's equation

$$-\frac{\partial u}{\partial x} = R_0 \cdot i + L_0 \cdot \frac{\partial i}{\partial t} \qquad -\frac{\partial i}{\partial x} = G_0 \cdot u + C_0 \cdot \frac{\partial u}{\partial t}$$

Assume R_0, L_0, G_0, C_0 are all constants

$$-\frac{\partial^2 u}{\partial x^2} = R_0 \cdot \frac{\partial i}{\partial x} + L_0 \cdot \frac{\partial^2 i}{\partial t \cdot \partial x} \qquad -\frac{\partial^2 i}{\partial t \cdot \partial x} = G_0 \cdot \frac{\partial u}{\partial t} + C_0 \cdot \frac{\partial^2 u}{\partial t^2}$$

$$-\frac{\partial^2 u}{\partial x^2} = R_0 \cdot \left(-G_0 \cdot u - C_0 \cdot \frac{\partial u}{\partial t} \right) + L_0 \cdot \left(-G_0 \cdot \frac{\partial u}{\partial t} - C_0 \cdot \frac{\partial^2 u}{\partial t^2} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = R_0 \cdot G_0 \cdot u + (R_0 \cdot C_0 + G_0 \cdot L_0) \cdot \frac{\partial u}{\partial t} + L_0 \cdot C_0 \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 i}{\partial x^2} = R_0 \cdot G_0 \cdot i + (R_0 \cdot C_0 + G_0 \cdot L_0) \cdot \frac{\partial i}{\partial t} + L_0 \cdot C_0 \cdot \frac{\partial^2 i}{\partial t^2}$$

Wave Equations for Lossless Transmission Line

If the line is lossless, then $R_0 = G_0 = 0$, therefore:

$$\frac{\partial^2 i}{\partial x^2} = L_0 \cdot C_0 \cdot \frac{\partial^2 i}{\partial t^2} = \frac{1}{v^2} \cdot \frac{\partial^2 i}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$v = \frac{1}{\sqrt{L_0 \cdot C_0}}$$

The velocity of the propagating wave

Conclusion: The voltage and the current travel along the transmission line are governed by the wave equations

Solutions of the Wave Equation for Loss less Transmission Line

$$u(x,t) = u_d\left(t - \frac{x}{v}\right) + u_i\left(t + \frac{x}{v}\right)$$ **Eq.1**

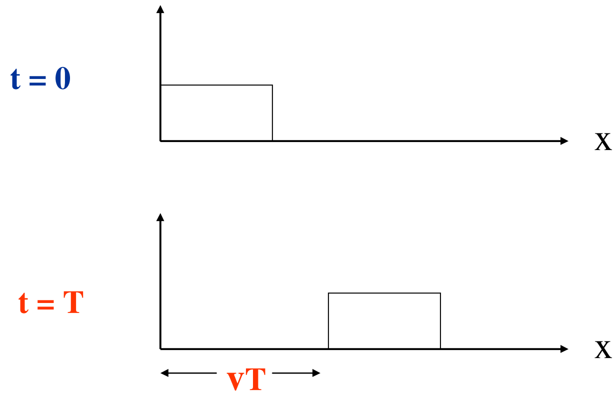
$$i(x,t) = i_d\left(t - \frac{x}{v}\right) + i_i\left(t + \frac{x}{v}\right)$$ **Eq.2**

$u_d\left(t - \frac{x}{v}\right)$ *and* $i_d\left(t - \frac{z}{v}\right)$ **Forward propagating wave**

$u_i\left(t + \frac{x}{v}\right)$ *and* $i_i\left(t + \frac{x}{v}\right)$ **Backward propagating wave**

Conclusion: The total voltage and the current are sum of the forward and the backward propagating waves along the transmission line as expressed in (1-2).

Example: Forward Propagating Wave



Derivation of the Power Equation

Multiply eq. (1) by i(x,t) and eq. (2) by u(x,t), add:

(1) $$-\frac{\partial u}{\partial x} = R_0 \cdot i + L_0 \cdot \frac{\partial i}{\partial t}$$

(2) $$-\frac{\partial i}{\partial x} = G_0 \cdot u + C_0 \cdot \frac{\partial u}{\partial t}$$

$$-\frac{\partial u}{\partial x} \cdot i = R_0 \cdot i^2 + L_0 \cdot i \cdot \frac{\partial i}{\partial t}$$

$$-\frac{\partial i}{\partial x} \cdot u = G_0 \cdot u^2 + C_0 \cdot u \cdot \frac{\partial u}{\partial t}$$

$$-\frac{\partial}{\partial x}(ui) = R_0 \cdot i^2 + G_0 \cdot u^2 + \frac{\partial}{\partial t}\left(\frac{L_0 \cdot i^2}{2}\right) + \frac{\partial}{\partial t}\left(\frac{C_0 \cdot u^2}{2}\right)$$

$$-\frac{\partial p}{\partial x} = p_d + \frac{\partial W}{\partial t}$$

$p = u \cdot i$ **Total Power**

$p_d = R_0 \cdot i^2 + G_0 \cdot u^2$ **Power Dissipation**

$W = \frac{1}{2} \cdot \left(L_0 \cdot i^2 + C_0 \cdot u^2\right)$ **Stored EM Energy**

Net Power Flow = Power Dissipation + Change in Stored Energy!

Steady-State TL Equations

$$u(x,t) = U(x)\sqrt{2}\cdot \sin\left(\omega t + \gamma_u(x)\right)$$

$$u(x,t) \Leftrightarrow \underline{U}(x) = U(x)\cdot e^{j\cdot\gamma_u}$$

$$i(x,t) = I(x)\sqrt{2}\cdot \sin\left(\omega t + \gamma_i(x)\right)$$

$$i(x,t) \Leftrightarrow \underline{I}(x) = I(x)\cdot e^{j\cdot\gamma_i}$$

Real functions of x and t

Complex functions of x only

ω being the frequency of the sinusoidal signal

Furthermore:

$$\frac{\partial}{\partial x}u(x,t) \quad \text{or} \quad i(x,t) \leftrightarrow \frac{d\underline{U}(x)}{dx} \quad \text{or} \quad \frac{d\underline{I}(x)}{dx}$$

$$\frac{\partial}{\partial t}u(x,t) \quad \text{or} \quad i(x,t) \leftrightarrow j\omega\ \underline{U}(x) \quad \text{or} \quad \underline{I}(x)$$

$$-\frac{\partial u}{\partial x} = R_0\cdot i + L_0\cdot\frac{\partial i}{\partial t} \Rightarrow -\frac{d\underline{U}}{dx} = R_0\cdot\underline{I} + j\omega L_0\cdot\underline{I} = (R_0 + j\omega L_0)\cdot\underline{I}$$

$$-\frac{\partial i}{\partial x} = G_0\cdot u + C_0\cdot\frac{\partial u}{\partial t} \Rightarrow -\frac{d\underline{I}}{dx} = (G_0 + j\omega C_0)\cdot\underline{U}$$

Comments on the steady-state TL equations

- Both equations are **o**rdinary **d**ifferential **e**quations (ODE) governing the complex voltage and the current $\underline{U}$(x) and $\underline{I}$(x) along the TL.
- The complex $\underline{U}$(x) and $\underline{I}$(x) have no direct physical meaning. On the other hand, once the complex $\underline{U}$ and $\underline{I}$ are known, the physical time-dependent, real solutions can be obtained.

Procedure for Sinusoidal Steady State Solutions

Convert to Complex Domain → Solve Complex Equations → Convert to Real Domain

Steady-state TL equations (cont)

$$-\frac{d\underline{U}}{dx} = (R_0 + j\omega L_0)\cdot \underline{I} \quad (4)$$

$$-\frac{d\underline{I}}{dx} = (G_0 + j\omega C_0)\cdot \underline{U} \quad (5)$$

If R_0, L_0, G_0, C_0 are all constants of x (homogeneous TL), then by taking derivative of (4) and making use of (5), we have:

$$\frac{d\underline{U}^2}{dx^2} = (R_0 + j\omega L_0)\cdot(G_0 + j\omega C_0)\cdot \underline{U} \Rightarrow \frac{d\underline{U}^2}{dx^2} - \underline{\gamma}^2 \cdot \underline{U} = 0$$

Similarly, we have:

$$\frac{d\underline{I}^2}{dx^2} = (R_0 + j\omega L_0)\cdot(G_0 + j\omega C_0)\cdot \underline{I} \Rightarrow \frac{d\underline{I}^2}{dx^2} - \underline{\gamma}^2 \cdot \underline{I} = 0$$

$$\underline{\gamma} = \sqrt{(R_0 + j\omega L_0)\cdot(G_0 + j\omega C_0)}$$

Complex propagation constant in 1/m

$$\underline{\gamma} = \sqrt{(R_0 + j\omega L_0)\cdot(G_0 + j\omega C_0)} = \alpha + j\beta$$

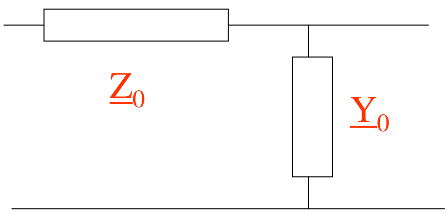
α = attenuation constant

$[\alpha]_{SI} = \frac{1 Neper}{1m} = \frac{1Np}{1m}$

β = phase constant

$[\beta] = \frac{1rad}{1m}$

$$\underline{Z}_0 = R_0 + j\omega L_0 \quad ,longitudinal \quad impedance$$
$$\underline{Y}_0 = G_0 + j\omega C_0 \quad ,transversal \quad admitances$$



The expression for the attenuation and phase constants with respect with the primary TL parameters

$$\alpha = \sqrt{\frac{1}{2}\left(Z_0 Y_0 + R_0 G_0 + \omega^2 L_0 C_0 \right)}$$

$$\beta = \sqrt{\frac{1}{2}\left(Z_0 Y_0 + \omega^2 L_0 C_0 - R_0 G_0 \right)}$$

Where Z_0 and Y_0 are:

$$Z_0 = \sqrt{R_0^2 + \omega^2 L_0^{\,2}}, \quad Y_0 = \sqrt{G_0^2 + \omega^2 C_0^{\,2}}$$

General Solutions

$$\frac{d\underline{U}^2}{dx^2} - \gamma^2 \cdot \underline{U} = 0$$

$$\underline{U}(x) = \underline{A} \cdot e^{-\underline{\gamma} \cdot x} + \underline{B} \cdot e^{+\underline{\gamma} \cdot x}$$

$$\underline{U}(x) = \underline{U}_d + \underline{U}_i$$

$$-\frac{d\underline{U}}{dx} = (R_0 + j\omega L_0) \cdot \underline{I} = \underline{A} \cdot \underline{\gamma} \cdot e^{-\underline{\gamma} \cdot x} - \underline{B} \cdot \underline{\gamma} \cdot e^{+\underline{\gamma} \cdot x}$$

$$\underline{I} = \frac{1}{(R_0 + j\omega L_0)} \cdot \left(\underline{A} \cdot \underline{\gamma} \cdot e^{-\underline{\gamma} \cdot x} - \underline{B} \cdot \underline{\gamma} \cdot e^{+\underline{\gamma} \cdot x} \right)$$

$$\underline{I} = \frac{\underline{\gamma}}{(R_0 + j\omega L_0)} \cdot (\underline{U}_d - \underline{U}_i) = \frac{1}{\underline{Z}_c} \cdot (\underline{U}_d - \underline{U}_i) = \underline{I}_d - \underline{I}_i$$

Physical Meaning of the General Solution:

- the total voltage/current are sum of the forward (exp[- γx]) and the backward (exp[+γx]) propagating waves:

$$\underline{U}(x) = \underline{U}_d + \underline{U}_i$$

- the forward propagating voltage (or current) waves $\underline{U}_d$ (or $\underline{I}_d$) are also called the direct voltage (or current) waves;
- the backward propagating voltage (or current) waves $\underline{U}_i$ (or $\underline{I}_i$) are also called the inverse voltage (or current) waves;

Relationship between $\underline{U}_d$, $\underline{U}_i$ and $\underline{I}_d$, $\underline{I}_i$

$$\frac{\underline{U}_d}{\underline{I}_d} = -\frac{\underline{U}_i}{\underline{I}_i} = \sqrt{\frac{R_0 + j\omega L_0}{G_0 + j\omega\omega_0}} = \underline{Z}_c$$

$\underline{Z}_c$: the characteristic impedance of the transmission line (Ω)

The secondary TL parameters

Complex propagation constant

$$\underline{\gamma} = \sqrt{(R_0 + j\omega L_0)\cdot(G_0 + j\omega C_0)}$$

Characteristic impedance (Ω)

$$\underline{Z}_c = \sqrt{\frac{R_0 + j\omega L_0}{G_0 + j\omega\omega_0}} = \sqrt{\frac{\underline{Z}_0}{\underline{Y}_0}}$$

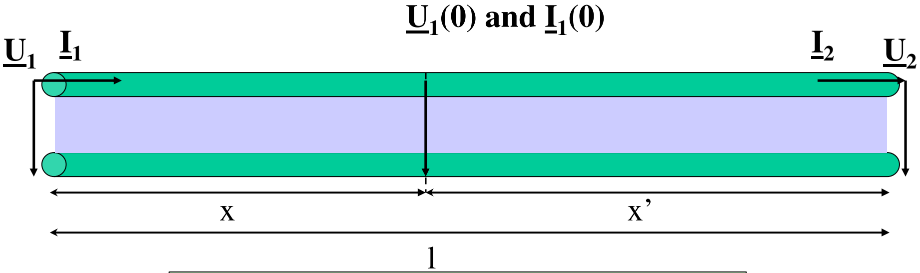
Where:
- $\underline{Z}_0$ is the longitudinal impedance
- $\underline{Y}_0$ is the transversal admittance

$$\underline{U}(x) = \underline{A}\cdot e^{-\underline{\gamma}\cdot x} + \underline{B}\cdot e^{+\underline{\gamma}\cdot x}$$

Only two knowns $\underline{A}$ and $\underline{B}$ remain to be determined!

Equivalent TL equations

1) Suppose that the voltage and the current for x = 0 are known: $\underline{U}_1(0)$ and $\underline{I}_1(0)$



$$\underline{U}(0) = \underline{A} \cdot e^{-\underline{\gamma} \cdot x} + \underline{B} \cdot e^{+\underline{\gamma} \cdot x} = \underline{A} + \underline{B} = \underline{U}_1$$

$$\underline{I}(0) = \frac{1}{\underline{Z}_c} \cdot \left(\underline{A} \cdot e^{-\underline{\gamma} \cdot x} - \underline{B} \cdot e^{+\underline{\gamma} \cdot x} \right) = \frac{1}{\underline{Z}_c} \cdot \left(\underline{A} - \underline{B} \right) = \underline{I}_1$$

$$\underline{A} + \underline{B} = \underline{U}_1$$

$$\underline{A} - \underline{B} = \underline{Z}_c \cdot \underline{I}_1$$

$$\Rightarrow \quad \underline{A} = \frac{\underline{U}_1 + \underline{I}_1 \cdot \underline{Z}_c}{2} \qquad \underline{B} = \frac{\underline{U}_1 - \underline{I}_1 \cdot \underline{Z}_c}{2}$$

Finally:

$$\underline{U}(x) = \left(\frac{\underline{U}_1 + \underline{I}_1 \cdot \underline{Z}_c}{2} \right) \cdot e^{-\underline{\gamma} \cdot x} + \left(\frac{\underline{U}_1 - \underline{I}_1 \cdot \underline{Z}_c}{2} \right) \cdot e^{+\underline{\gamma} \cdot x}$$

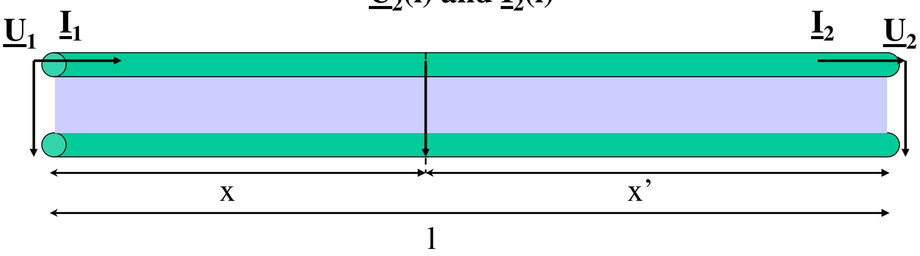
$$\underline{I}(x) = \frac{1}{\underline{Z}_c} \left\{ \left(\frac{\underline{U}_1 + \underline{I}_1 \cdot \underline{Z}_c}{2} \right) \cdot e^{-\underline{\gamma} \cdot x} - \left(\frac{\underline{U}_1 - \underline{I}_1 \cdot \underline{Z}_c}{2} \right) \cdot e^{+\underline{\gamma} \cdot x} \right\}$$

$$\underline{U}(x) = \underline{U}_1 \cdot ch(\underline{\gamma} \cdot x) - \underline{I}_1 \cdot \underline{Z}_c \cdot sh(\underline{\gamma} \cdot x)$$

$$\underline{I}(x) = \underline{I}_1 \cdot ch(\underline{\gamma} \cdot x) - \underline{U}_1 \cdot \frac{1}{\underline{Z}_c} \cdot sh(\underline{\gamma} \cdot x)$$

2) Suppose that the voltage and the current for x = l are known

$\underline{U}_2$(l) and $\underline{I}_2$(l)

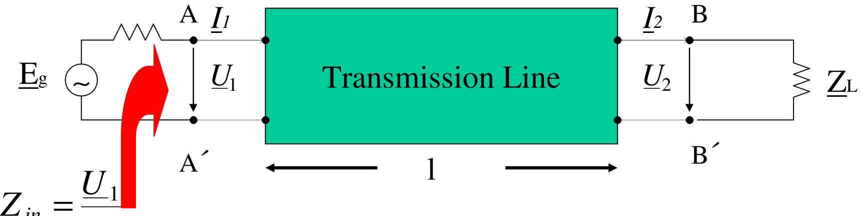


$$\underline{U}(x') = \underline{U}_2 \cdot ch(\underline{\gamma} \cdot x') + \underline{I}_2 \cdot \underline{Z}_c \cdot sh(\underline{\gamma} \cdot x')$$

$$\underline{I}(x') = \underline{I}_2 \cdot ch(\underline{\gamma} \cdot x') + \underline{U}_2 \cdot \frac{1}{\underline{Z}_c} \cdot sh(\underline{\gamma} \cdot x')$$

Very important equations !!

Inpute impedance of a TL



$$\underline{Z}_{in} = \frac{\underline{U}_1}{\underline{I}_1}$$

$$\underline{Z}_{in}(x'=l)=\frac{\underline{U}_1}{\underline{I}_1}=\frac{\underline{U}_2\cdot ch(\underline{\gamma}\cdot l)+\underline{I}_2\cdot\underline{Z}_C\cdot sh(\underline{\gamma}\cdot l)}{\underline{I}_2\cdot ch(\underline{\gamma}\cdot l)+\underline{U}_2\cdot\frac{1}{\underline{Z}_C}\cdot sh(\underline{\gamma}\cdot l)}$$

Supposing the external impedance: $\underline{Z}_L=\frac{\underline{U}_2}{\underline{I}_2}$

$$\underline{Z}_{in}=\frac{\underline{Z}_L\cdot ch(\underline{\gamma}\cdot l)+\underline{Z}_C\cdot sh(\underline{\gamma}\cdot l)}{ch(\underline{\gamma}\cdot l)+\frac{\underline{Z}_L}{\underline{Z}_C}\cdot sh(\underline{\gamma}\cdot l)}=\underline{Z}_C\cdot\frac{\underline{Z}_L\cdot ch(\underline{\gamma}\cdot l)+\underline{Z}_C\cdot sh(\underline{\gamma}\cdot l)}{\underline{Z}_C\cdot ch(\underline{\gamma}\cdot l)+\underline{Z}_L\cdot sh(\underline{\gamma}\cdot l)}$$

Particular cases for the input impedances:

a) TL in short circuit: $\underline{Z}_L\to 0$

$$\underline{Z}^0{}_{in}=\underline{Z}_C\cdot\frac{sh(\underline{\gamma}\cdot l)}{ch(\underline{\gamma}\cdot l)}=\underline{Z}_C\cdot th(\underline{\gamma}\cdot l)$$

b) TL in open circuit: $\underline{Z}_L\to\infty$

$$\underline{Z}^{sc}{}_{in}=\underline{Z}_C\cdot\frac{ch(\underline{\gamma}\cdot l)}{sh(\underline{\gamma}\cdot l)}=\underline{Z}_C\cdot\frac{1}{th(\underline{\gamma}\cdot l)}$$

If we take into account both particular cases:

$$\underline{Z}^0{}_{in}\cdot\underline{Z}^{sc}{}_{in}=\underline{Z}_C^2\Rightarrow\underline{Z}_C=\sqrt{\underline{Z}^0{}_{in}\cdot\underline{Z}^{sc}{}_{in}}$$

$$\frac{\underline{Z}^0{}_{in}}{\underline{Z}^{sc}{}_{in}}=(th(\underline{\gamma}\cdot l))^2\Rightarrow\underline{\gamma}\cdot l=arg\,th\sqrt{\frac{\underline{Z}^0{}_{in}}{\underline{Z}^{sc}{}_{in}}}$$

$$\underline{\gamma}=\alpha+\beta j=\frac{1}{l}\cdot arg\,th\sqrt{\frac{\underline{Z}^0{}_{in}}{\underline{Z}^{sc}{}_{in}}}$$

c) TL closed on the characteristic impedance: $\underline{Z}_L=\underline{Z}_C$

$$\underline{Z}_{in}=\underline{Z}_C\cdot\frac{\underline{Z}_C\cdot ch(\underline{\gamma}\cdot l)+\underline{Z}_C\cdot sh(\underline{\gamma}\cdot l)}{\underline{Z}_C\cdot ch(\underline{\gamma}\cdot l)+\underline{Z}_C\cdot sh(\underline{\gamma}\cdot l)}=\underline{Z}_C$$

Lossless Transmission Line

If the line is lossless, then $R_0 = G_0 = 0$, therefore:

$$\underline{Z}_c = \sqrt{\frac{R_0 + j\omega L_0}{G_0 + j\omega C_0}} = \sqrt{\frac{L_0}{C_0}} = real \ (\ pur \ \ resitive \ \ )$$

$$\underline{\gamma} = \sqrt{(R_0 + j\omega L_0)\cdot(G_0 + j\omega C_0)} = j\omega\sqrt{L_0 \cdot C_0}$$

$$\underline{\gamma} = \alpha + j\beta = j\omega\sqrt{L_0 \cdot C_0}$$

$$\alpha = 0$$

$$\beta = \omega\sqrt{L_0 \cdot C_0} = 2\pi f\sqrt{L_0 \cdot C_0} = \frac{2\pi}{v}\cdot f = \frac{2\pi}{v}\cdot\frac{1}{T} = \frac{2\pi}{\lambda}$$

Where v is the velocity of the wave: $v = \dfrac{1}{\sqrt{L_0 \cdot C_0}}$

The inpute impedance of a losseless TL

$$\underline{Z}in = \underline{Z}_C \cdot \frac{\underline{Z}_L \cdot ch(\underline{\gamma}\cdot l) + \underline{Z}_C \cdot sh(\underline{\gamma}\cdot l)}{\underline{Z}_C \cdot ch(\underline{\gamma}\cdot l) + \underline{Z}_L \cdot sh(\underline{\gamma}\cdot l)}$$

$$ch(\underline{\gamma}\cdot l) = ch(j\cdot\beta\cdot l) = cos(\beta\cdot l)$$
$$sh(\underline{\gamma}\cdot l) = sh(j\cdot\beta\cdot l) = j\cdot sin(\beta\cdot l)$$

$$\underline{Z}_{in} = \sqrt{\frac{L_0}{C_0}}\cdot\frac{\underline{Z}_L \cdot \cos(\beta\cdot l) + \underline{Z}_C \cdot j\cdot\sin(\beta\cdot l)}{\underline{Z}_C \cdot \cos(\beta\cdot l) + \underline{Z}_L \cdot j\cdot\sin(\beta\cdot l)}$$

Particular cases:

a) The length of the TL is equal with a quarter of the wave length:

$$l = \frac{\lambda}{4}$$

$$\underline{Z}_{in} = \sqrt{\frac{L_0}{C_0}} \cdot \frac{\underline{Z}_L \cdot \cos(\beta \cdot \frac{\lambda}{4}) + \underline{Z}_C \cdot j \cdot \sin(\beta \cdot \frac{\lambda}{4})}{\underline{Z}_C \cdot \cos(\beta \cdot \frac{\lambda}{4}) + \underline{Z}_L \cdot j \cdot \sin(\beta \cdot \frac{\lambda}{4})}$$

$$\beta = \omega\sqrt{L_0 \cdot C_0} = 2\pi f \sqrt{L_0 \cdot C_0} = \frac{2\pi}{v} \cdot \frac{1}{T} = \frac{2\pi}{\lambda} \Rightarrow \boxed{\beta \cdot \lambda = 2\pi}$$

$$\underline{Z}_{in} = \sqrt{\frac{L_0}{C_0}} \cdot \frac{\underline{Z}_L \cdot \cos(\frac{\pi}{2}) + \underline{Z}_C \cdot j \cdot \sin(\frac{\pi}{2})}{\underline{Z}_C \cdot \cos(\frac{\pi}{2}) + \underline{Z}_L \cdot j \cdot \sin(\frac{\pi}{2})} = \sqrt{\frac{L_0}{C_0}} \cdot \frac{\underline{Z}_C \cdot j}{\underline{Z}_L \cdot j} = \frac{L_0}{C_0} \cdot \frac{1}{Z_L}$$

If, $\underline{Z}_L$ = inductive $\underline{Z}_{in}$ will be capacitive and

$\underline{Z}_L$ = capacitive $\underline{Z}_{in}$ will be inductive

- **So, that a losselessTL having the length equal with a quarter of the wave length will act as a impedance transformer**

Particular cases:

b) The length of the TL is equal with a half of the wave length:

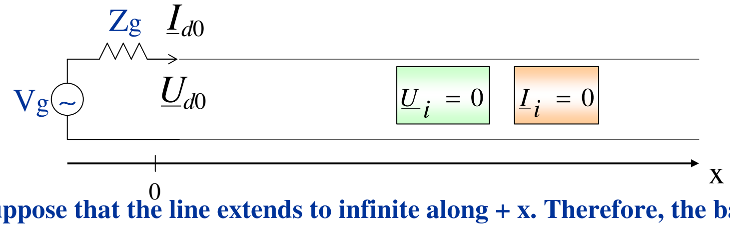
$$\boxed{l = \frac{\lambda}{2}}$$

$$\underline{Z}_{in} = \sqrt{\frac{L_0}{C_0}} \cdot \frac{\underline{Z}_L \cdot \cos(\beta \cdot l) + \underline{Z}_C \cdot j \cdot \sin(\beta \cdot l)}{\underline{Z}_C \cdot \cos(\beta \cdot l) + \underline{Z}_L \cdot j \cdot \sin(\beta \cdot l)} \qquad \boxed{\beta \cdot \lambda = 2\pi}$$

$$\underline{Z}_{in} = \sqrt{\frac{L_0}{C_0}} \cdot \frac{\underline{Z}_L \cdot \cos(\pi) + \underline{Z}_C \cdot j \cdot \sin(\pi)}{\underline{Z}_C \cdot \cos(\pi) + \underline{Z}_L \cdot j \cdot \sin(\pi)} = \sqrt{\frac{L_0}{C_0}} \cdot \frac{\underline{Z}_L}{\underline{Z}_C} = \underline{Z}_L$$

- **So, that a losselessTL having the length equal with a half of the wave length will act as a impedance adaptor**

Semi-Infinite Line



Suppose that the line extends to infinite along + x. Therefore, the backward propagating waves do not exist. We obtain:

$$\underline{U}(x) = \underline{U}_d + \underline{U}_i \quad\Rightarrow\quad \underline{U}(x) = \underline{U}_d = \underline{A}\cdot e^{-\underline{\gamma}\cdot x}$$

$$\underline{I} = \frac{1}{\underline{Z}_c}\cdot(\underline{U}_d - \underline{U}_i) = \underline{I}_d - \underline{I}_i \quad\Rightarrow\quad \underline{I}_d = \frac{\underline{U}_d}{\underline{Z}_c}$$

γ is complex so that:

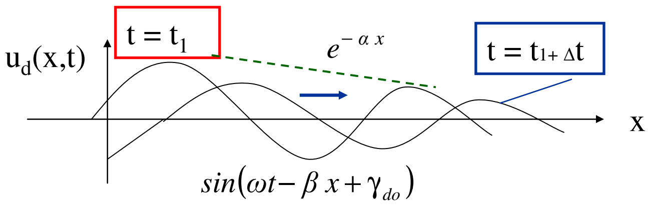
$$\underline{U}_d(x) = \underline{U}_{d0}\cdot e^{-\underline{\gamma}\cdot x} = U_{d0}\cdot e^{j\cdot\gamma_{d0}}\cdot e^{-\alpha\cdot x - j\cdot\beta\cdot x}$$

$$\underline{U}_d(x) = \underline{U}_{d0}\cdot e^{-\underline{\gamma}\cdot x} = U_{d0}\cdot e^{j\cdot\gamma_{d0}}\cdot e^{-\alpha\cdot x - j\cdot\beta\cdot x}$$

The instantaneous direct voltage and current are

$$u_d(x,t) = U_{d0}\cdot\sqrt{2}\cdot e^{-\alpha\cdot x}\cdot sin(\omega t - \beta\cdot x + \gamma_{d0})$$

$$i_d(x,t) = I_{d0}\cdot\sqrt{2}\cdot e^{-\alpha\cdot x}\cdot sin(\omega t - \beta\cdot x + \gamma_{d0} + \vartheta)$$



Amplitude: decay according to exp($-\alpha$x), therefore α is the amplitude decay constant of the wave

The instantaneous inverse voltage and current:

$$u_i(x,t) = U_{i0} \cdot \sqrt{2} \cdot e^{\alpha \cdot x} \cdot sin(\omega t + \beta \cdot x + \gamma_{i0})$$

$$i_i(x,t) = I_{i0} \cdot \sqrt{2} \cdot e^{\alpha \cdot x} \cdot sin(\omega t + \beta \cdot x + \gamma_{i0} + \vartheta i)$$

$$\frac{d}{dt}(\omega t + \beta x + \gamma_{i0}) = 0 \Rightarrow v_i = -\frac{dx}{dt} = -\frac{\omega}{\beta} = v_d$$

Amplitude: decay according to exp(αx), therefore α is the amplitude decay constant of the wave

Phase Velocity

Definition: The velocity at which the phase of the wave travels

Let us observe a point of constant phase such that:

$$\omega t - \beta x = const$$

Take derivative with respect to time t: $\omega - \beta \frac{dx}{dt} = 0$

Re-arrange so that that phase velocity is expressed as:

$$v_d = \frac{dx}{dt} = \frac{\omega}{\beta}$$

- which is a function of frequency and dependent on the physical properties of the transmission line through β.

Heaviside TL conditions

If: $\frac{R_0}{L_0} = \frac{G_0}{C_0}$ $\Rightarrow$ $\underline{\gamma} = \sqrt{(R_0 + j\omega L_0)\cdot(G_0 + j\omega C_0)}$

$$\underline{\gamma} = \sqrt{R_0 \cdot G_0 \cdot \left(1 + \frac{j\omega L_0}{R_0}\right)\cdot\left(1 + \frac{j\omega C_0}{G_0}\right)} = \sqrt{R_0 \cdot G_0 \cdot \left(1 + \frac{j\omega L_0}{R_0}\right)^2} =$$

$$= \left(1 + \frac{j\omega L_0}{R_0}\right)\cdot\sqrt{R_0 \cdot G_0} = \sqrt{R_0 \cdot G_0} + j\omega L_0 \cdot \sqrt{\frac{G_0}{R_0}} = \sqrt{R_0 \cdot G_0} + j\omega\sqrt{L_0 \cdot C_0}$$

$$\alpha = \sqrt{R_0 \cdot G_0}$$
$$\beta = \omega\sqrt{L_0 \cdot C_0}$$

Important !!!!

Heaviside TL conditions

$$\alpha = \sqrt{R_0 \cdot G_0}$$
$$\beta = \omega\sqrt{L_0 \cdot C_0}$$

$$v_d = \frac{\omega}{\beta}$$

$$v_d = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{L_0 \cdot C_0}} = \frac{1}{\sqrt{L_0 \cdot C_0}} \quad (\textit{independent of } f)$$

TL without distortions !!!!!