The Theory of Electromagnetic Field

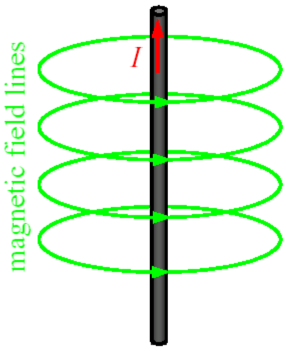
Chapter 3

Magnetostatics fields

Magnetostatics fields

Introduction notes

- Hans Christian Oersted discovered (in 1819) the magnetic force of a current carrying wire
- The right-hand rule of current direction and magnetic field direction
- André Marie Ampère measures the magnetic force (1820-1825) on a wire with current
- Jean-Baptiste Biot and Felix Savart (1825) measured accurately the magnetic field of a current element

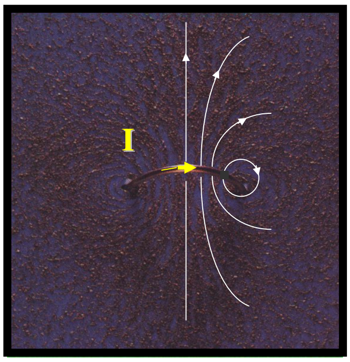


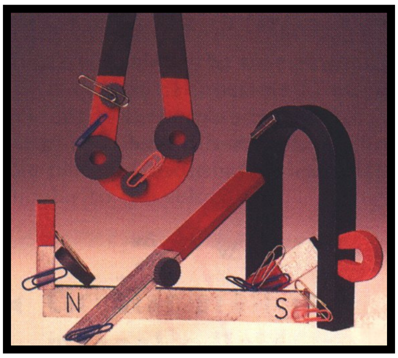
Magnetostatics fields

- Our most familiar experience of magnetism is through **permanent magnets.**
- These are made of materials which exhibit a property we call **"ferromagnetism"** - i.e., they can be magnetized.
- Depending on how we position two magnets, they will attract or repel, ***i.e.* they exert forces on each other.**
- Thus, a magnet must have an associated field: a **magnetic field.**
- But we have not been able, so far, to isolate a magnetic monopole (the equivalent of an electric charge).
- We describe magnets as having two **magnetic poles**: North (**N**) and South (**S**).
- They also attract un-magnetized ferromagnetic materials.
- We can map out the field of a magnet using either a small magnet or small magnetic materials....

Magnetostatics fields

Magnetic Fields (B)





Currents

Permanent Magnets

Magnetostatics fields

- The origin of magnetism lies in moving electric charges. Moving (or rotating) charges generate magnetic fields.
- An electric current generates a magnetic field.
- A magnetic field will exert a force on a moving charge.
- A magnetic field will exert a force on a conductor that carries an electric current.

Magnetostatics fields

- **Stationary charge:**
  - $v_q = 0$
  - $E \neq 0 \qquad B = 0$

A stationary charge produces an electric field only.

- **Moving charge:**
  - $v_q \neq 0$ and $v_q = constant$
  - $E \neq 0 \qquad B \neq 0$

A uniformly moving charge produces an electric and magnetic field.

- **Accelerating charge:**
  - $v_q \neq 0$ and $a_q \neq 0$
  - $E \neq 0 \qquad B \neq 0 \qquad$ *Radiating field*

A accelerating charge produces an electric and magnetic field and a radiating electromagnetic field.

Magnetostatics fields

Units and definitions:

$$1T = 10^4 G$$

Tesla (SI unit) — Gauss

$$1T = 1\frac{Wb}{m^2}$$

Weber

$\vec{B}$ — Magnetic field vector, Magnetic induction, Magnetic flux density

$\vec{H}$ — Magnetic field strength

$$\vec{B} = \mu \cdot \vec{H}$$

Magnetostatics fields

Permeability

$$\mu = \mu_r \cdot \mu_o$$

- μ_o → Permeability of free space
- μ_r → Relative permeability for a medium
- μ → Permeability of the medium

$$\mu_o = 4 \cdot \pi \cdot 10^{-7} \left\{ \frac{H}{m} \right\}$$

Exact constant

$$\left\{ \frac{H}{m} \right\} \Leftrightarrow \left\{ \frac{Wb}{m} \right\}$$

Magnetostatics fields

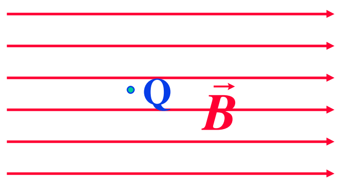
Relative permeability μ_r and susceptibility χ_m

	μ_r	χ_m	
Bismuth	0.99983	-1.66 E-4	Diamagnetic
Mercury	0.999968	-3.2 E-5	Diamagnetic
Gold	0.999964	-3.6 E-5	Diamagnetic
Silver	0.99998	-2.60 E-5	Diamagnetic
Lead	0.999983	-1.7 E-5	Diamagnetic
Copper	0.999991	-0.98 E-5	Diamagnetic
Water	0.999991	-0.88 E-5	Diamagnetic
Vacuum	1.000	0	
Air	1.00000036	3.6 E-7	Paramagnetic
Aluminium	1.000021	2.5 E-5	Paramagnetic
Palladium	1.00082	8.2 E-4	Paramagnetic
Cobalt	250	---	Ferromagnetic
Nickel	600	---	Ferromagnetic
Iron	6000	---	Ferromagnetic

Magnetostatics fields

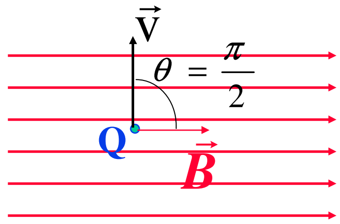
Magnetic forces

1. Lorentz force



If the charge is not moving with respect to the magnetic field (or if the charge moves parallel to the field).

The force is zero.



If the charge is moving, there is a force on the charge, ***perpendicular*** to both **v** and **B**.

$$\vec{F} = Q\,\vec{v} \times \vec{B}$$

Magnetostatics fields

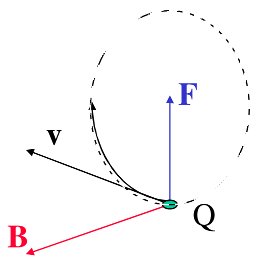
- As we saw, force is perpendicular to both **v** and **B**.
- The force is also largest for **v** perpendicular to **B**, smallest for **v** parallel to **B**.

This can be summarized as:

$$\vec{F} = Q \cdot \vec{v} \times \vec{B}$$ or: $$F = Q\,v\,B\,\sin\theta$$

Units of Magnetic Field

$$B = \frac{F}{Q\,v\,\sin\theta}$$



Therefore the units of magnetic field are:

$$\frac{1N \cdot 1s}{1C \cdot 1m} = 1T \qquad (Tesla)$$

(Note: 1 Tesla = 10,000 Gauss)

Magnetostatics fields

The **magnetic force** is ***different*** from the **electric force**.

Whereas the electric force acts in the same direction as the electric field:

$$\vec{F} = Q \cdot \vec{E}$$

The magnetic force acts in a direction orthogonal to the magnetic field:

$$\vec{F} = Q \cdot \vec{v} \times \vec{B}$$

(Use "Right-Hand" rule to determine direction of F)

If a magnetic field and an electric field are simultaneously present, their *forces* obey the superposition principle and may be added **vectorially**:

$$\vec{F} = Q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right)$$

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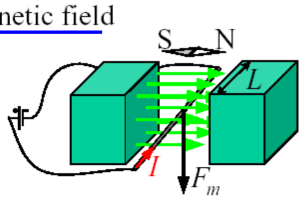
2. Force on a current in a static magnetic field

The force magnitude is measured to give: $F_m = ILB$, N

and the direction is as shown. The factor B is related to the strength of the magnetic field of the permanent magnet.



Experiment 1

It was shown that the force magnitude is proportional to the sine of the angle between the wire and the magnetic field direction. Finally, in vector form: $\vec{F}_m = I(\vec{L} \times \vec{B})$, N

The force does not depend on the position, which is to be expected from a large flat field source (remember the infinite charged sheet).

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The above law is referred to as Ampere's force law (or motor equation). It is also valid for the force exerted by a constant magnetic field B on a *current element* $I \cdot d\vec{s}$

$$d\vec{F}_m = I \cdot \left(d\vec{s} \times \vec{B}\right), \; N$$

The concept of *current element* is essential in the magnetic and EM theory. It plays a role similar to that of a point charge in electrostatics. The current element is an infinitesimally short current carrier (e.g. piece of wire). It is specified by both, its current I and its line element $d\vec{s}$. The direction of the line element is that of the current.

It can be shown that this law is directly related to the force with which the magnetic field acts upon a charge Q moving with a velocity $\vec{v}$.

Magnetostatics fields

The current element can be expressed via the current density J,

$$I \cdot d\vec{s} = \vec{J} \cdot d\vec{A} \cdot d\vec{s} = \vec{J} \cdot dv, \; Am$$

$$d\vec{F}_m = \left(\vec{J} \times \vec{B}\right) \cdot dv, \quad N$$

We already known how to express the current density via the charge density and velocity,

$$\vec{J} = \rho_v \cdot \vec{v}, \quad A/m^2$$

$$d\vec{F}_m = \left(\rho_v \cdot dv\right)\left(\vec{v} \times \vec{B}\right)$$

For any moving charge Q in a magnetic field, the magnetic force is proportional to the amount of charge, to its velocity and to the strength of the magnetic field:

$$d\vec{F}_m = Q \cdot \left(\vec{v} \times \vec{B}\right)$$

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If one adds to that magnetic force the electrostatic force, one arrives at the fundamental force equation in electromagnetics: the Lorentz' force equation:

$$\vec{F} = \vec{F} + \vec{F}_m = Q \cdot \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

So far, the following has been established:

- The field of a permanent magnet affects wires with current.
- The field of a permanent magnet does not affect charged wires.
- The direction of the magnetic field and the current is important: the force is proportional to their cross product.
- The force is proportional to the length of the current element.

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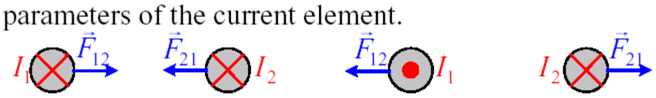
3. Ampere's force law between two wires with current

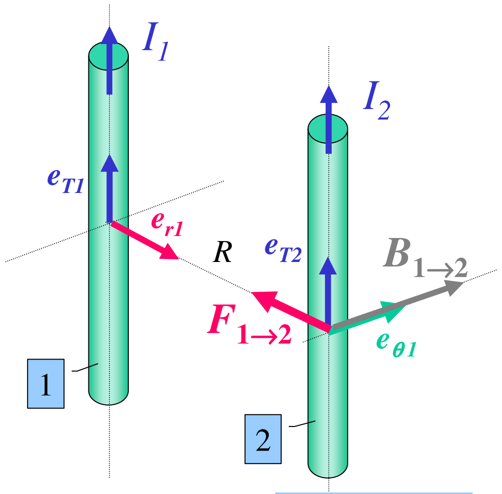
The next step in the study of magnetic fields was to establish whether currents themselves can create the force field observed with permanent magnets, and how this field would depend on the parameters of the current element.



Experiment 2

In 1820, Ampère measured the forces between two parallel current-carrying wires of length *L*. He established that when the currents had the same direction, the wires attracted each other. On the contrary, when the currents were in opposite directions, the wires repelled each other. The force magnitude was measured to be:

Magnetostatics fields



Question: determine $F_{1\to2}$

Relations:

$$\vec{B} = \frac{\mu_0 I}{2\pi R}\mathbf{e}_\theta \; ; \; \mathbf{e}_\theta = \mathbf{e}_T \times \mathbf{e}_r$$

$$d\vec{F}_L = I \cdot d\vec{s} \times \vec{B}; \; d\vec{s} = ds \cdot \mathbf{e}_T$$

Calculation

$$\mathbf{B}_{1\to2} = \frac{\mu_0 I_1}{2\pi R}\mathbf{e}_{\theta1}$$

$$\boldsymbol{F}_{1\to2} = \int I_2 . \boldsymbol{e}_{T2} \times \boldsymbol{B}_{1\to2} . dl_2$$

$$\mathbf{F}_{1\to2} = \frac{\mu_0 I_1 I_2}{2\pi R} L_2(-\mathbf{e}_{r1})$$

$$\boldsymbol{e}_{T2} \times \boldsymbol{e}_{\theta1} = -\boldsymbol{e}_{r1}$$

If $L_1 = L_2$: $\boldsymbol{F}_{1\to2} = -\boldsymbol{F}_{2\to1}$

Magnetostatics fields

$$|\vec{F}| = k_m \frac{I^2 L}{R}, \quad \text{N}$$

When the currents of the wires were different, the force was measured to be:

$$|\vec{F}| = k_m \frac{I_1 I_2 L}{R}, \quad \text{N}$$

In SI, the magnetic constant k_m is: $k_m = \frac{\mu_0}{2\pi}$

This measurement is used to the define the electric current I standard (1 ampere) in the SI system.

The force depends on the magnetic properties of the region surrounding the wires, and is much stronger in the presence of ferromagnetic materials: $\mu = \mu_r \mu_0$, H/m

$\mu_0 = 4\pi \times 10^{-7}$ H/m

Magnetostatics fields

The force per unit length on either conductor is:

$$F_1 = \frac{F}{L} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{R}, \quad \text{N/m}$$

This is a typical 2-D field behaviour (compare with the 2-D Coulomb's law).

$$F_1 = \frac{1}{2\pi\varepsilon_0}\frac{\rho_{l_1}\rho_{l_2}}{R}$$

From *Experiment* 2, three important conclusions follow.

- Steady currents are sources of magnetic field (the same field that is produced by permanent magnets).
- If the current-carrying wires are long enough, the magnetic force has the typical for any other field 2-D behaviour ($\sim 1/R$).
- The magnetic field of a long wire with current can now be derived.

Magnetostatics fields

In *Experiment* 1, it was shown that a current element of length L in a permanent magnetic field B of a magnet experiences a force:

$$\vec{F} = I\vec{L}\times\vec{B}, \quad \text{N}$$

where $\vec{B}$ points from the south pole to the north pole of the magnet. The force exerted on wire 1 by the current flowing in wire 2 was shown to be in *Experiment* 2:

$$F_{12} = \frac{\mu}{2\pi}\frac{I_1 I_2 L}{R}, \quad \text{N}$$

On order both experiments to be consistent, the following must be true for $\vec{B}$:

- it is orthogonal to the wire element; and
- $|\vec{B}_2| = \frac{\mu}{2\pi}\frac{I_2}{R}, \quad \text{T} = \frac{\text{N}}{\text{A}\times\text{m}} = \frac{\text{H}\times\text{A}}{\text{m}^2}$ ← Magnetic flux density

Magnetostatics fields

Biot-Savart (Laplace) formula

Biot-Savart (Laplace) formula is the fundamental method in magnetostatics, analogous to Coulomb's formula in electrostatics.

It was founded from measurements of the torque on a magnetic needle (1820).

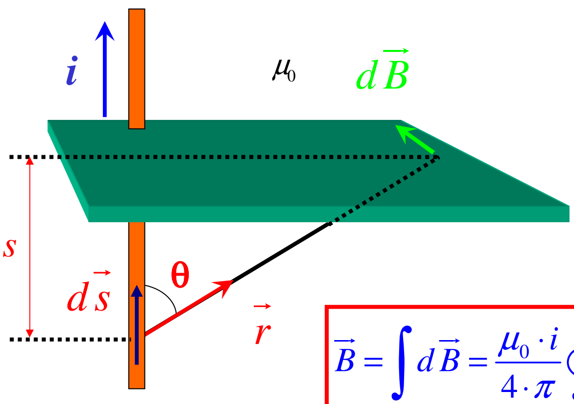
It gives means to calculate the magnetic field at any point of space from a known distribution of currents.

We need to add up the bits of magnetic field ***dB*** arising from each infinitesimal length *ds*.

Supposed that the structure consists in:

- i is the current in the wire (element) [Amps (A)]
- $d\vec{s}$ is the length of the current element measured in the direction of current flow (m)
- $\vec{r}$ is the distance from the element to the point of interest (m)

Magnetostatics fields



$$dB = \frac{\mu_0}{4\cdot\pi}\cdot\frac{i\cdot ds\cdot\sin\theta}{r^2} \Rightarrow$$

$$d\vec{B} = \frac{\mu_0}{4\cdot\pi}\cdot\frac{i\cdot\left(d\vec{s}\times\vec{r}\right)}{r^3}$$

$$\vec{B} = \int_C d\vec{B} = \frac{\mu_0\cdot i}{4\cdot\pi}\oint_C \frac{d\vec{s}\times\vec{r}}{r^3}$$

$\mu_0 = 4\pi\times10^{-7}\ \frac{H}{m}$ is called the permeability of the free space.

Magnetostatics fields

Magnetic field computation using Biot-Savart-Laplace formula

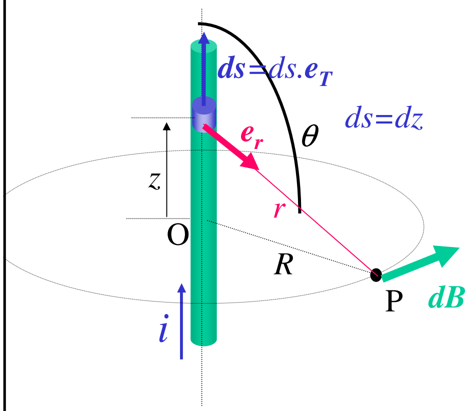
Algorithm of computation:

- Analysis and symmetry
- Approach to solution
  - consider the infinitesimal current element *ds*
  - consider the vector *r* measured from the infinitesimal current element to the computation point
  - the magnetic field ***dB*** arising from each infinitesimal length *ds* is considered.
- Calculations
- Conclusions

Magnetostatics fields

Example 1

Consider a long straight wire carrying a current i. We want to find the magnetic field **B** at a point P, a distance R from the wire.



Biot & Savart: $d\vec{B} = \frac{\mu_0 i}{4\pi}\frac{d\vec{s}\times\vec{r}}{r^3} = \frac{\mu_0 i}{4\pi}\frac{\vec{e}_T \times \vec{e}_r}{r^2} ds$

Approach: Current line elements *ds*

Calculation: $\boldsymbol{e_T} \, x \, \boldsymbol{e_r} = \boldsymbol{e_\phi}$;
tangential component only: ***dB***

$$dB = \frac{\mu_0 i}{4\pi}\frac{\sin\theta}{r^2} dz$$

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$$dB = \frac{\mu_0 i}{4\pi} \cdot \frac{1}{R^2+z^2} \cdot \frac{R}{\sqrt{R^2+z^2}}\, dz$$

$$B = \oint_C dB = \frac{\mu_0 i}{4\pi} \cdot \oint_C \frac{1}{R^2+z^2} \cdot \frac{R}{\sqrt{R^2+z^2}}\, dz$$

$$B = \frac{\mu_0 i}{4\pi} \cdot \int_{-\infty}^{\infty} \frac{R}{\left(R^2+z^2\right)^{\frac{3}{2}}} \cdot dz = \frac{\mu_0 \cdot i}{4 \cdot \pi \cdot R} \cdot \left(\frac{z}{\sqrt{R^2+z^2}} \right) \Bigg|_{-\infty}^{\infty} = \frac{\mu_0 \cdot i}{2 \cdot \pi \cdot R}$$

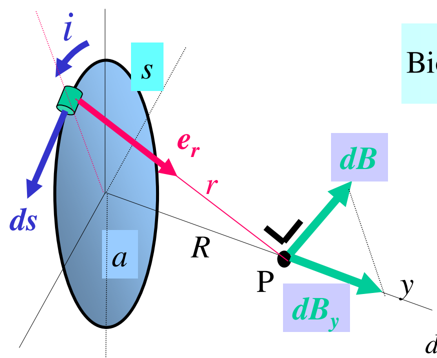
The final result is:

$$\vec{B}_P = \frac{\mu_0 \cdot i}{2 \cdot \pi \cdot R}\, \vec{e}_\varphi$$

Example 2

Consider a circular circuit with radius a, carrying a current i. We want to find the magnetic field **B** at a point P, a distance R from the wire.

Magnetostatics fields



$ds = ds.\, e_T$

Biot & Savart: $d\vec{B} = \frac{\mu_0 \cdot i}{4 \cdot \pi} \frac{\mathbf{e_T} \times \mathbf{e_r}}{r^2} ds$

Question: Determine ***B*** in P

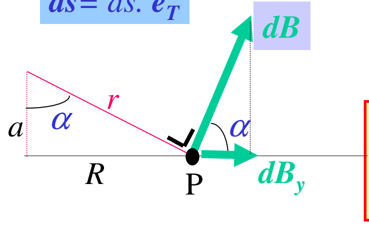
Approach: Current line elements *ds*

Calculation: $e_T \, x \, e_r = 1$;
symmetry: ***y-*** component only:

$$dB_y = dB \cdot \cos\alpha = \frac{\mu_0 \cdot i}{4 \cdot \pi} \frac{1}{r^2} ds \cos\alpha$$

$$B_y = \oint_C dB_y = \frac{\mu_0 \cdot i}{4 \cdot \pi} \cdot \frac{1}{a^2+R^2} \cdot 2\pi a \cdot \frac{a}{\sqrt{a^2+R^2}}$$

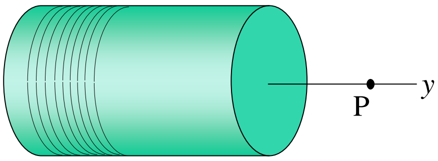
r^2 $\oint_C ds$ $\cos\alpha$



$$\vec{B}_P = \frac{\mu_0 \cdot i \cdot a^2}{2\left(a^2+R^2\right)^{3/2}} \cdot \vec{e}_y$$

Magnetostatics fields

Example 3. Magnetic field of a circular solenoid



Radius: a ; Current i
Length: L
Coils: N , or per meter: n

Question: Determine $\boldsymbol{B}$ in P

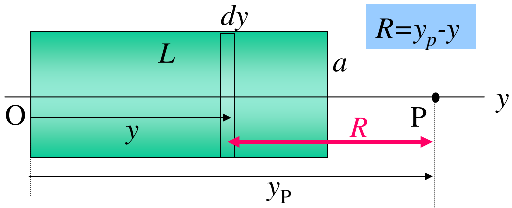
Approach: Solenoid = set of circular circuits ;
and for each circuit:

$$\vec{B}_P = \frac{\mu_0 \cdot i \cdot a^2}{2 \cdot \left(a^2 + R^2\right)^{3/2}} \cdot \vec{e}_y$$

R is distance from circuit to P

Each circuit: strip dy; current $di = n.dy.i$

$R = y_p - y$



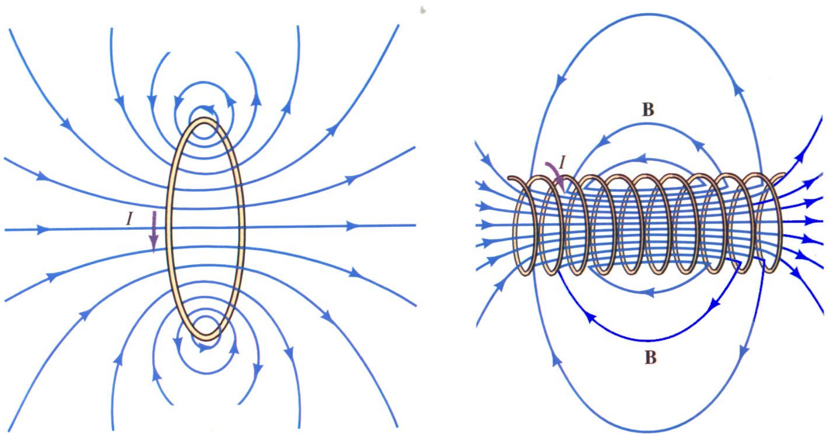
$$\vec{B} = \int_0^L \frac{\mu_0 \cdot ni \cdot dy \cdot R^2}{2 \cdot \left(R^2 + a^2\right)^{3/2}} \vec{e}_y$$

Result for $L \to \infty$:

$$\boldsymbol{B} = \mu_0\, n\, i\, \boldsymbol{e}_y$$

Result independent of a, L

Magnetostatics fields



Current loop

Coil or solenoid

Magnetostatics fields

Ampere's law

When we studied the electrostatic field, Coulomb's law was the first step only, which gave the relation between the field's vector and its sources. Later on, we derived the electrostatic field divergence and curl, or in equivalent integral terms, its circulation and net flux:

$$\nabla \times \vec{E} = 0 \Leftrightarrow \oint_C \vec{E} d\vec{l} = 0 \qquad \nabla \cdot \vec{D} = \rho \Leftrightarrow \oiint_S \vec{D} \cdot d\vec{s} = Q$$

The above equations are true for any electrostatic field regardless of its sources.

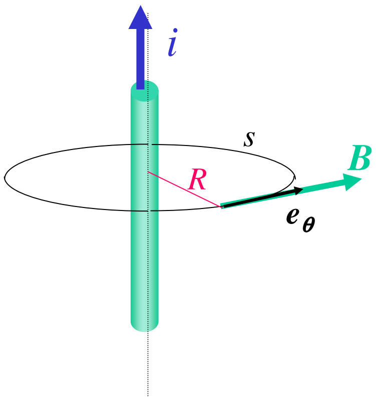
Ampère's law defines the circulation (and the curl) of the magnetic field.

Magnetostatics fields

It was already shown that the magnetic flux density created by a straight wire with current is:

$$\vec{B} = \frac{\mu_0 \cdot i}{2 \cdot \pi \cdot R} \vec{e}_\varphi$$

If we now integrate around a path of radius R enclosing the wire, we will obtain the circulation of B:



$$\oint_C \vec{B} \cdot d\vec{s} = \oint_C \frac{\mu_0 i}{2\pi R} \vec{e}_\varphi \cdot d\vec{s} =$$
$$= \oint_C \frac{\mu_0 i}{2\pi R} \vec{e}_\varphi \cdot ds \cdot \vec{e}_\varphi = \frac{\mu_0 i}{2\pi R} \oint_C ds =$$
$$= \frac{\mu_0 i}{2\pi R} \cdot 2\pi R = \mu_0 \cdot i$$

Magnetostatics fields

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \cdot i$$

We are now introducing a vector, which will be independent of the medium, and will be entirely defined by the sources:

The ***magnetic field intensity*** or simply the ***magnetic field vector*** is:

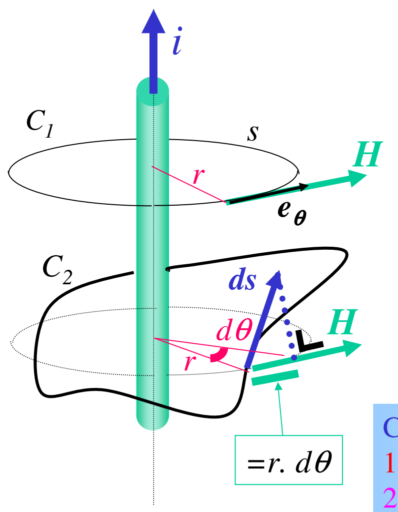
$$\vec{H} = \frac{\vec{B}}{\mu_0}, \quad \frac{1A}{1m}$$

$$\oint_C \vec{H} \cdot d\vec{s} = i$$

The above is Ampère's law, which is valid not only for straight wires but for any current distribution and any contour of integration. The general form of Ampère's law can be derived from Biot-Savart law in a robust way. The two laws imply each other, as in electrostatics, Coulomb's law and Gauss' law are closely related.

The current in the right-hand side of Ampère's law is the total net current flowing through any open surface bounded by the contour *C*.

Magnetostatics fields



Determine the "circulation of ***H*** -field" along an arbitrary circuit *c*

$$\oint_{C_1} \vec{H} \cdot d\vec{s} = \oint_{C_2} \vec{H} \cdot d\vec{s} = i$$

Consequences:

1. More currents through C_2 add up;
2. Currents outside C_2 do not contribute;
3. Position of current inside C_2 is not important.

Magnetostatics fields

Ampere's law can be derived in its differential form as follows:

$$\oint_{C_1} \vec{H} \cdot d\vec{s} = i \Rightarrow \oint_{C} \vec{H} \cdot d\vec{s} = \iint_{S_C} \vec{J} \cdot d\vec{A}$$

$$\iint_{S_C} rot\vec{H} \cdot d\vec{A} = \iint_{S_C} \vec{J} \cdot d\vec{A} \Rightarrow rot\vec{H} = \vec{J}$$

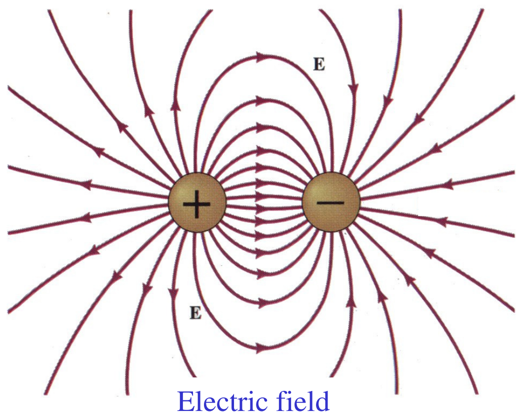
It is now clear that the electric currents are the curl-sources of the magnetic filed.

Notice that the magnetic field has a non-zero curl, and this non-zero curl equals the electric current density, unlike the electrostatic field, which is a curl-free field.

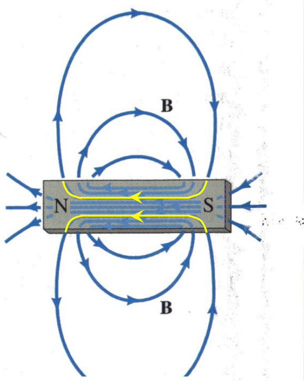
$rot\vec{H} = \vec{J}$ $\qquad$ $rot\overline{E} = 0$

Magnetostatics fields

Irrotational Field $\qquad$ **Rotational Field**



Electric field



B- field

$rot\overline{E} = 0$ $\qquad$ $rot\vec{H} = \vec{J}$

Electromagnetic Field

Note: until now we have considered that the surface S is immobile.

In the most general case of media in movement (surface has a relative speed towards the media), the Ampere's law in differential form must be completed as:

$$rot\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} + \rho_v \cdot \vec{v} + rot\left(\vec{D} \times \vec{v}\right) \quad , \quad A/m^2$$

Conduction current density

Displacement current density

Convection current density

Roentgen current density

From the physical point of view, the correction current appears due to the displacement, with the speed v (towards the surface S_C), of bodies charged with charges (having the volume density of the charge ρ_v) and the Roentgen current appears due to the displacement with the speed v of polarized bodies (having the volumetric density of the polarized charge).

Electromagnetic Field

$$rot\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} + \rho_v \cdot \vec{v} + rot\left(\vec{D} \times \vec{v}\right)$$

$$\oint_{C(S)} \vec{H} \cdot d\vec{s} = \iint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} + \rho_v \cdot \vec{v} + rot\left(\vec{D} \times \vec{v}\right) \right) \cdot d\vec{A}$$

$$\oint_{C(S)} \vec{H} \cdot d\vec{s} = i + i_D + i_C + i_R$$

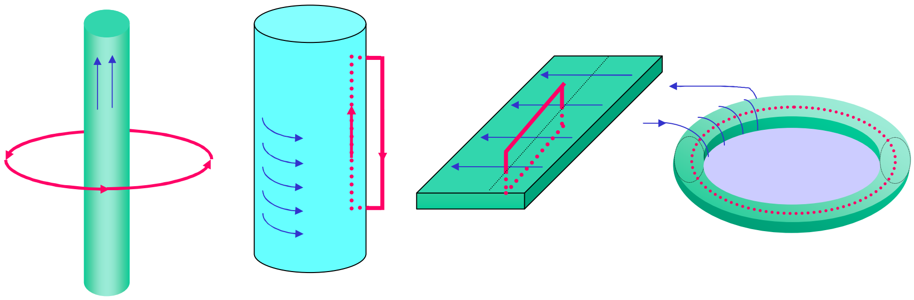
Ampere's integral form

In the most general case, a magnetic field can be produced by:

- conduction currents
- displacement currents
- convection currents
- Roentgen currents

Magnetostatics fields

Symmetries for Ampere's Law



Wire, ∞ long	Solenoid, ∞ long	Plane, ∞ extending	Toroide, along core line

Magnetostatics fields

Example 1

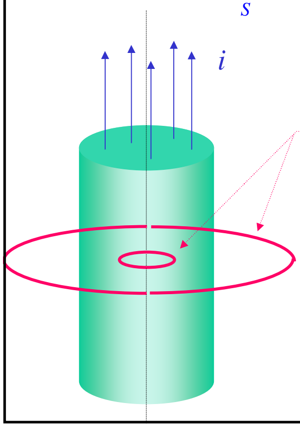
Ampere law used for the computation of H-field from a thick wire

<u>Cylinder:</u> radius a

current : $i = \iint_S \vec{J} \cdot \vec{dA}$

$$\oint_C \vec{H} \cdot d\vec{s} = i = \iint_S \vec{J} \cdot d\vec{A}$$

Options for current:
I: at surface
II: in volume (suppose: homogeneous)

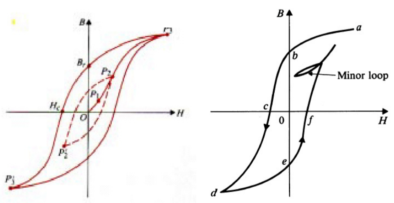


Use Ampere-circuits (radius r):

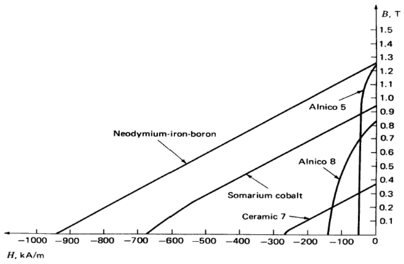
$r \geq a$	$H(r) \cdot 2\pi r = i$	$H(r) = \frac{i}{2\pi r}$
$r \leq a$	$(II): H(r) \cdot 2\pi r = i \frac{\pi r^2}{\pi a^2}$	$H(r) = \frac{i \cdot r}{2\pi a^2}$

Magnetostatics fields

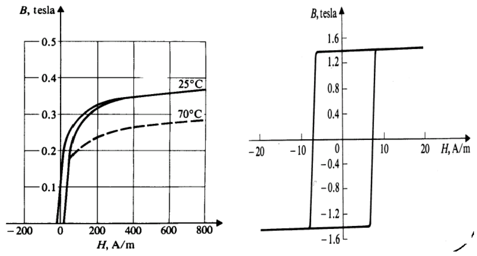
Magnetic properties of materials



Magnetic Hysteresis



Demagnetization curves of permanent magnets



Hysteresis loops of soft Ferrites & recording media

Magnetostatics fields

Laws of magnetostatics

The magnetic flux law

Definition:

The total magnetic flux through a closed surface is always zero.

$$\oiint_{\Sigma} \vec{B} \cdot d\vec{A} = 0$$

Integral form of the law

⇩

$$\oiint_{\Sigma} \vec{B} \cdot d\vec{A} = \iiint_{V_\Sigma} div\,\vec{B} \cdot dv = 0$$

⇩

$$div\,\vec{B} = 0$$

Differential form of the law

$$div_s\,\vec{B} = \bar{n}_{12} \cdot \left(\vec{B}_2 - \vec{B}_1\right) = B_{2n} - B_{1n} = 0$$

Boundary conditions

Magnetostatics fields

Laws of electrostatics

$$\oiint_{\Sigma} \vec{D} \cdot d\vec{A} = q_{\Sigma}$$

Integral form of the law

$$\oiint_{\Sigma} \vec{D} \cdot d\vec{A} = \iiint_{V_{\Sigma}} div\vec{D} \cdot dv = q_{\Sigma}$$

$$div\vec{D} = \rho_v$$

Differential form of the law

$$div_s\vec{D} = \bar{n}_{12} \cdot \left(\vec{D}_2 - \vec{D}_1\right) = D_{2n} - D_{1n} = \rho_s$$

Boundary conditions

Magnetostatics fields

Laws of magnetostatics

The magnetization law

From experimental studies, it is found that the magnetization vector (the temporary component) is strongly related to the magnetic field strength. For most common magnetic materials, these two vectors are collinear and proportional for a wide range of values of H (linear materials and isotropic).

$$\vec{M}_t = \chi_m \cdot \vec{H}$$

Valid just for linear materials

where: χ_m is the magnetic susceptibility of the material.

When the magnetic susceptibility depends on the magnetic field strength H, it is said that the **medium is nonlinear**, because all the field relations become nonlinear equations. When the magnetic susceptibility depends on the position in the volume of the magnetic body, it is said that the problem is **inhomogeneous**, as opposed to the homogeneous case when the properties of the material are constant throughout the volume. Moreover, the magnetic properties may depend on the direction of the applied field. This is called **anisotropy** of the magnetic material.

Magnetostatics fields

Relative permeability μ_r and susceptibility χ_m

Material	μ_r	χ_m	Class
Bismuth	0.99983	-1.66 E-4	Diamagnetic
Mercury	0.999968	-3.2 E-5	Diamagnetic
Gold	0.999964	-3.6 E-5	Diamagnetic
Silver	0.99998	-2.60 E-5	Diamagnetic
Lead	0.999983	-1.7 E-5	Diamagnetic
Copper	0.999991	-0.98 E-5	Diamagnetic
Water	0.999991	-0.88 E-5	Diamagnetic
Vacuum	1.000	0	
Air	1.00000036	3.6 E-7	Paramagnetic
Aluminium	1.000021	2.5 E-5	Paramagnetic
Palladium	1.00082	8.2 E-4	Paramagnetic
Cobalt	250	---	Ferromagnetic
Nickel	600	---	Ferromagnetic
Iron	6000	---	Ferromagnetic

Magnetostatics fields

Laws of electrostatics

The polarization law

From experimental studies, it is found that the polarization vector (the temporary component) is strongly related to the electric vector field. For most common dielectrics, these two vectors are collinear and proportional for a wide range of values of E (linear materials).

$$\vec{P}_t = \chi_e \cdot \varepsilon_0 \cdot \vec{E}$$ Valid just for linear materials

where: χ_e is the electric susceptibility of the material.

Magnetostatics fields

Laws of magnetostatics

The relation between B, H and M vectors

The vector sum, between the magnetization vectors (both components) and the magnetic field strength, multiplied with the permeability of the vacuum, is equal, at any moment and point, with the magnetic flux density:

$$\vec{B} = \mu_0 \cdot \left(\vec{H} + \vec{M}_t + \vec{M}_p\right)$$

For materials without permanent magnetization: $\vec{B} = \mu_0 \cdot \left(\vec{H} + \vec{M}_t\right)$

For linear materials without permanent magnetization:

$$\vec{B} = \mu_0 \cdot \left(\vec{H} + \vec{M}_t\right) = \mu_0 \cdot \left(\vec{H} + \chi_m \cdot \vec{H}\right) = \mu_0 \cdot (1 + \chi_m) \cdot \vec{H} = \mu \cdot \vec{H}$$

For materials with anisotropy and without permanent magnetization:

$$\vec{B} = \overline{\overline{\mu}} \cdot \vec{H}$$

Magnetostatics fields

Laws of magnetostatics

The relation between B, H and M vectors

Then, the relation between the magnetic flux vector and the magnetic field strength is a tensor one:

$$\vec{B} = \overline{\overline{\mu}} \cdot \vec{H} \quad \Rightarrow \quad \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Fortunately, it is often suffices to assume that the medium is **homogeneous**, **linear** and **isotropic**. This is the simplest possible case.

Final note on the physical meaning of the relative magnetic permeability: it shows how many times the magnetic field strength is decreased or increased in the volume of the magnetic material due to the effect of the magnetization.

Note: In general the magnetization vector consists in 2 components:
- a temporary component (M_t) and a permanent one (M_p)

Magnetostatics fields

Laws of electrostatics

The vector sum between polarizations (both components) and the electrical field intensity, multiplied with the permitivity of the vacuum, is equal, at any moment and point, with the electrical flux density:

$$\vec{D} = \varepsilon_0 \cdot \vec{E} + \vec{P}_t + \vec{P}_p$$

For materials without permanent polarization: $\vec{D} = \varepsilon_0 \cdot \vec{E} + \vec{P}_t$

For linear materials without permanent polarization: $\vec{D} = \varepsilon \cdot \vec{E}$

For materials with anisotropy and without permanent polarization:

$$\vec{D} = \overline{\overline{\varepsilon}} \cdot \vec{E}$$

Magnetostatics fields

Laplace and Poisson equations in electrostatic

$$div(-gradV) = \frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = \Delta V = -\frac{\rho_v}{\varepsilon}$$

Free space – Laplace equation

$\rho_v = 0$

Materials – Poisson equation

$\rho_v \neq 0$

$$V = \frac{1}{4\pi\varepsilon} \int_V \frac{\rho_v}{r} dv$$

Magnetostatics fields

Poisson's equation in magnetostatics

$div\,\vec{B} = 0$ → $\vec{B} = rot\,\vec{A}$ → $\vec{H} = \frac{\vec{B}}{\mu} = \frac{1}{\mu} rot\,\vec{A}$

A = magnetic vector potential

Ampere's law in differential form $rot\,\vec{H} = \vec{J}$ → $rot\left(\frac{1}{\mu} \cdot rot\,\vec{A}\right) = \vec{J}$

If the medium is linear, homogeneous and isotropic:

$$rot\left(rot\,\vec{A}\right) = \mu \cdot \vec{J}$$

$$rot\left(rot\,\vec{A}\right) = grad\left(div\,\vec{A}\right) - \Delta\vec{A} = \mu \cdot \vec{J}$$

Imposing the Coulomb condition: $div\,\vec{A} = 0$

Magnetostatics fields

Poisson's equation

$grad\left(div\,\vec{A}\right) - \Delta\vec{A} = \mu \cdot \vec{J}$ → $\vec{\nabla}^2 \vec{A} = \Delta\vec{A} = -\mu \cdot \vec{J}$

$div\,\vec{A} = 0$

Vector Poisson's Equation

Scalar Laplacian

$$\vec{\nabla}^2 \vec{A} = \nabla^2 A_x \cdot \vec{i} + \nabla^2 A_y \cdot \vec{j} + \nabla^2 A_z \cdot \vec{k}$$

$$\vec{J} = J_x \cdot \vec{i} + J_y \cdot \vec{j} + J_z \cdot \vec{k}$$

$$\nabla^2 A_x = -\mu \cdot J_x$$
$$\nabla^2 A_y = -\mu \cdot J_y$$
$$\nabla^2 A_z = -\mu \cdot J_z$$

3 scalar Poisson's equations

Magnetostatics fields

Poisson's equation

Analogy with the Poisson's equation for electrostatic field

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

With the already known solution:

$$V = \frac{1}{4\cdot\pi\cdot\varepsilon}\int_{V_\Sigma}\frac{\rho_v}{r}\cdot dv$$

$$\nabla^2 A_x = -\mu\cdot J_x$$
$$\nabla^2 A_y = -\mu\cdot J_y$$
$$\nabla^2 A_z = -\mu\cdot J_z$$

$$A_x = \frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{J_x}{r}\cdot dv$$
$$A_y = \frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{J_y}{r}\cdot dv$$
$$A_z = \frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{J_z}{r}\cdot dv$$

Magnetostatics fields

Poisson's equation

$$\vec{A} = \left(\frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{J_x}{r}\cdot dv\right)\cdot\vec{i} + \left(\frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{J_y}{r}\cdot dv\right)\cdot\vec{j} + \left(\frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{J_z}{r}\cdot dv\right)\cdot\vec{k}$$

$$\vec{A} = \frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{\vec{J}}{r}\cdot dv$$

Vector magnetic potential solution for Poisson's equation for volume current distribution

If we consider a thin wire then:

$$J = \frac{i}{S}$$

where S is the transversal area of the wire and i the total current

$$\vec{A} = \frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{i}{S}\cdot\frac{dv}{r} = \frac{\mu}{4\cdot\pi}\int_{V_\Sigma}\frac{i}{\vec{S}}\cdot\frac{\vec{S}\cdot d\vec{s}}{r} = \frac{\mu\cdot i}{4\cdot\pi}\oint_C\frac{d\vec{s}}{r}$$

Magnetostatics fields

Poisson's equation

Vector magnetic potential solution for Poisson's equation valid only for thin wires:

Supposed that the structure consists in:

i is the current in the wire (element) [Amps (A)]

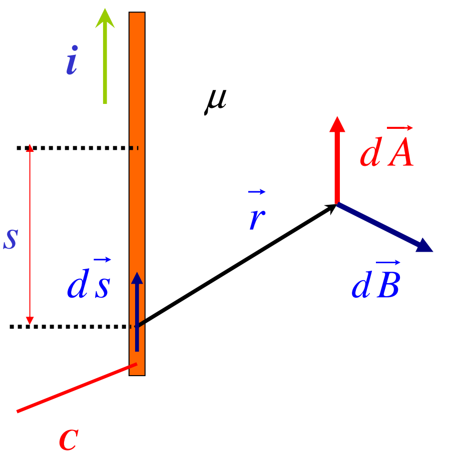
$d\vec{s}$ is the length of the current element measured in the direction of current flow (m)

$\vec{r}$ is the distance from the element to the point of interest (m)

$$\vec{A} = \frac{\mu \cdot i}{4 \cdot \pi} \oint_C \frac{d\vec{s}}{r}$$

Magnetostatics fields

Poisson's equation



$$\vec{A} = \frac{\mu \cdot i}{4 \cdot \pi} \oint_C \frac{d\vec{s}}{r}$$

If the vector magnetic potential is known then the Biot-Savart's formula can be deduced:

$$\vec{B} = rot\vec{A} = rot\left(\frac{\mu \cdot i}{4 \cdot \pi} \oint_C \frac{d\vec{s}}{r}\right) =$$

$$= \frac{\mu \cdot i}{4 \cdot \pi} \cdot rot\left(\oint_C \frac{d\vec{s}}{r}\right) = \frac{\mu \cdot i}{4 \cdot \pi} \oint_C \frac{d\vec{s} \times \vec{r}}{r^3}$$

Magnetostatics fields

Note:

$$\vec{B} = rot\,\vec{A} \Rightarrow \Psi = \iint_{Sc} \vec{B}\cdot d\vec{A} = \iint_{Sc} rot\,\vec{A}\cdot d\vec{A} = \oint_C \vec{A}\cdot d\vec{s}$$

In magnetostatics fields a flux is much easier to be computed (in comparison with electrostatics fields) !!! Instead of a surface integral it is enough to compute a line integral.

$$\vec{A_2} = \frac{\mu\cdot i}{4\cdot\pi}\oint_{C_2}\frac{d\vec{s_2}}{r}$$

Considering the magnetic vector potential for thin wires, the magnetic flux computed in a closed path C_1 is:

$$\Psi = \oint_{C_1}\vec{A_2}\cdot d\vec{s_1} = \oint_{C_1}\left(\frac{\mu\cdot i}{4\cdot\pi}\oint_{C_2}\frac{d\vec{s_2}}{r}\right)\cdot d\vec{s_1} =$$

$$= \frac{\mu\cdot i}{4\cdot\pi}\oint_{C_1}\oint_{C_2}\frac{d\vec{s_1}\cdot d\vec{s_2}}{r}$$

Magnetostatics fields

Note:

$$\vec{B} = rot\,\vec{A} \Rightarrow \Psi = \iint_{Sc} \vec{B}\cdot d\vec{A} = \iint_{Sc} rot\,\vec{A}\cdot d\vec{A} = \oint_C \vec{A}\cdot d\vec{s}$$

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$$= \frac{\mu\cdot i}{4\cdot\pi}\oint_{C_1}\oint_{C_2}\frac{d\vec{s_1}\cdot d\vec{s_2}}{r}$$

Magnetostatics fields

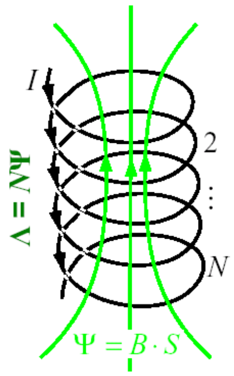
Analogy

Electrostatics	Magnetostatics
Scalar Potential - V	Vector Potential - A
E-field	H-field
Permitivity ε	Permeability μ
Volume Charge Density ρ_v	Current Density $\overline{J}$
Surface Charge Density ρ_s	Surface Current Density $\overline{J}_s$
Capacitance - C	Inductance - L
Laplace's Equation	Laplace's Equation
Poisson's Equation	Poisson's Equation

Magnetostatics fields

Flux linkage

In inductors, it is very important not only to achieve large magnetic flux but it is also important to make this flux flow through as many coils as possible. *The sum of all fluxes piercing the surfaces bounded by the turns of a coil is called flux linkage.*



In most inductors, the fluxes flowing through the coils are the same and the flux linkage is simply: $\Lambda = N\Psi, \quad \text{Wb}$

Thus, if the magnetic flux density is uniform inside the inductor (the coil), then the flux linkage is: $\Lambda = NBS, \quad \text{Wb}$

Magnetostatics fields

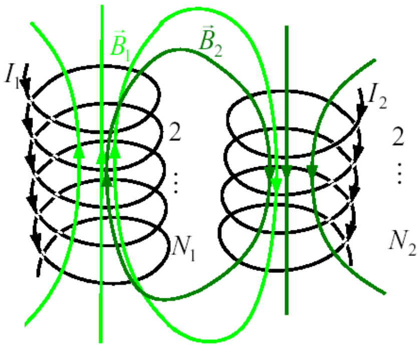
Flux linkage

The flux linkage may be self flux linkage and mutual flux linkage. A single coil has only its own self flux linkage, i.e. this is the flux created by its own current, which flows through its own turns.

The mutual flux linkage is defined **only** if a pair of magnetically coupled coils exists.

The mutual flux linkage of coil 1 is due to the magnetic field of the coil 2, which induced emf in coil 1.

$$\begin{cases}\Lambda_{11} = N_1\Psi_{11} \sim N_1^2 I_1 \\ \Lambda_{12} = N_1\Psi_{12} \sim N_1 N_2 I_2\end{cases} \quad \begin{cases}\Lambda_{22} = N_2\Psi_{22} \sim N_2^2 I_2 \\ \Lambda_{21} = N_2\Psi_{21} \sim N_2 N_1 I_1\end{cases}$$



$$\Lambda_1 = \Lambda_{11} + \Lambda_{12} \qquad \Lambda_2 = \Lambda_{22} + \Lambda_{21}$$

Magnetostatics fields

Self inductance and mutual inductance

It was already shown that the magnetic flux (and therefore the flux linkage) in solenoids and toroids is proportional to the current. The coefficient of proportionality between the flux linkage and the current is called *inductance L*:

$$\Lambda = LI, \quad \text{Wb=V}\times\text{s} \quad \Rightarrow \boxed{L = \frac{\Lambda}{I}}, \quad \text{H=}\frac{\text{Wb}}{\text{A}}=\frac{\text{V}\times\text{s}}{\text{A}}=\Omega\times\text{s}$$

Thus, in circuit theory: $= L\dfrac{dI}{dt}, \quad \text{V}$

The self inductance corresponds to the self flux linkage:

$$L_{11} = \frac{\Lambda_{11}}{I_1}, \quad \text{H}$$

The mutual inductance is defined through the mutual flux linkage:

$$M_{21} = M_{12} = \frac{\Lambda_{21}}{I_1} = \frac{\Lambda_{12}}{I_2}, \quad \text{H}$$

Magnetostatics fields

Self inductance and mutual inductance

It can be shown that the flux of any steady current is proportional to its current, and, therefore *its inductance is always a constant as long as the magnetic permeability is a constant.*

Consider a loop of current along a contour C_1. The flux through the open surface S_1 bounded by C_1 is

$$\Psi_{11} = \iint_{S_1} \vec{B}_1 \cdot d\vec{A}_1 , \quad Wb$$

From Biot-Savart's law, the magnetic flux density produced by this current contour at any point of space is:

$$\vec{B}_1 = \frac{\mu \cdot I_1}{4 \cdot \pi} \oint_{C_1} \frac{d\vec{s}_1 \times \vec{r}}{r^3}, \quad T$$

$$\Psi_{11} = \iint_{S_1} \left(\frac{\mu \cdot I_1}{4 \cdot \pi} \oint_{C_1} \frac{d\vec{s}_1 \times \vec{r}}{r^3} \right) \cdot d\vec{A} , \quad Wb \qquad \Psi_{11} \sim \mu \cdot I_1$$

Magnetostatics fields

Self inductance and mutual inductance

As long as the magnetic permeability μ does not depend on the current, the inductance is also a constant, which does not depend on the current *I*.

In ferromagnetic materials however the magnetic permeability depends on the strength of the field $|\vec{H}|$. Therefore, it depends on the current, too. Thus, the inductance of coils with ferromagnetic cores is a non-linear function of the current.

$$\Psi \sim \mu(I) I \Rightarrow \Psi = L(I) \cdot I, \quad \text{Wb}$$

In general, the self inductance of any contour *C* with current can be expressed via the MSF vectors as:

$$L = \frac{\Psi}{I} = \frac{\iint_{S[C]} \vec{B} \cdot d\vec{A}}{\oint_C \vec{H} \cdot d\vec{s}} = \mu \cdot \frac{\iint_{S[C]} \vec{H} \cdot d\vec{A}}{\oint_C \vec{H} \cdot d\vec{s}},$$

Magnetostatics fields

$$L = \mu \cdot \frac{\iint_{S[C]} \vec{H} \cdot d\vec{A}}{\oint_C \vec{H} \cdot d\vec{s}}, \quad [H]$$

$$C = \varepsilon \cdot \frac{\iint_{\Sigma} \vec{E} \cdot d\vec{A}}{\int_1^2 \vec{E} \cdot d\vec{s}}, \quad [F]$$

Just as it was shown that for capacitors: $\frac{C}{G} = \frac{\varepsilon}{\sigma}$

It can be shown that: $L_{ext} \cdot C = \mu \cdot \varepsilon$

Note: L_{ext} represents only the external inductance, related to the external flux of an inductor.

In an inductor such as a coaxial or parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the internal inductance L_{int} while that produced by the flux external to it is called external inductance. The total inductance L is:

$$L = L_{int} + L_{ext}$$

Magnetostatics fields

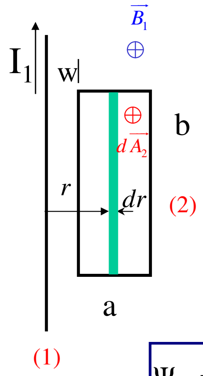
Typical examples of inductors are toroids, solenoids, coaxial transmission lines, and parallel-wire transmission lines. The inductance of each of these inductors can be determined by following a procedure similar to that taken in determining the capacitance of a capacitor.

For a given inductor, we find the self-inductance L by taking the following steps:

1. Choose a suitable coordinate system.
2. Let the inductor carry current I.
3. Determine *B* from Biot-Savart's law (or from Ampere's law if symmetry exists) and calculate Ψ from $\Psi = \iint_S \vec{B} \cdot d\vec{A}$
4. Finally find *L* from $L = \frac{\Lambda}{I} = \frac{N \cdot \Psi}{I}$

The mutual inductance between two circuits may be calculated by taking a similar procedure OR applying Neumann's formula.

Magnetostatics fields



Example no. 2

Find the mutual inductance between a long, straight wire (circuit 1) and a rectangular loop (a by b) placed at a distance w (circuit 2).

$$\oint_C \vec{H}_1 \cdot d\vec{s} = I_1 \Rightarrow 2 \cdot \pi \cdot r \cdot H_{1\varphi} = I_1$$

$$B_{1\varphi} = \mu \cdot \frac{I_1}{2 \cdot \pi \cdot r}, \quad T$$

$$\Psi_{12} = \iint_{S_2} \vec{B}_1 \cdot d\vec{A}_2 = \iint_{S_2} B_{1\varphi} \cdot dA_2 = \iint_{S_2} \mu \cdot \frac{I_1}{2 \cdot \pi \cdot r} \cdot dA_2 = \mu \cdot \frac{I_1}{2 \cdot \pi} \iint_{S_2} \frac{dA_2}{r}$$

$$\Psi_{12} = \mu \cdot \frac{I_1}{2 \cdot \pi} \iint_{S_2} \frac{dA_2}{r} = \mu \cdot \frac{I_1}{2 \cdot \pi} \int_w^{w+a} \frac{b \cdot dr}{r} = \mu \cdot \frac{I_1 \cdot b}{2 \cdot \pi} \cdot ln \frac{w+a}{w}$$

$$L_{12} = \frac{\Psi_{12}}{I_1} = \mu \cdot \frac{b}{2 \cdot \pi} ln \frac{w+a}{w}$$

Electromagnetic field

Introduction

So far we have:

$$div\vec{E} = \frac{\rho_v}{\varepsilon} \qquad rot\vec{E} = 0 \qquad div\vec{B} = 0 \qquad rot\vec{H} = \vec{J}$$

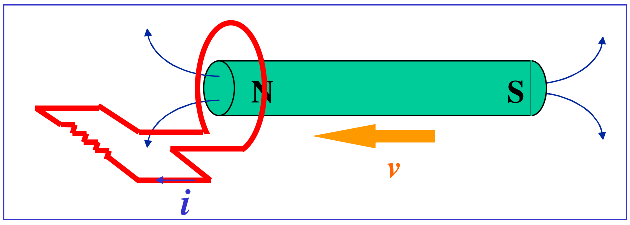
Electrostatic field ($div\vec{E} = \frac{\rho_v}{\varepsilon}$, $rot\vec{E} = 0$) — **Magnetostatic field** ($div\vec{B} = 0$, $rot\vec{H} = \vec{J}$)

These equations are OK for static fields, i.e. those fields independent of time. When fields vary as a function of time the curl equations acquire an additional term.

$rot\vec{E} = 0$ gets a $-\frac{\partial \vec{B}}{\partial t}$ $\qquad$ $rot\vec{H} = \vec{J}$ gets a $\frac{\partial \vec{D}}{\partial t}$

Electromagnetic field

Faraday’s Experiments



- Michael Faraday discovered induction in 1831.
- Moving the magnet induces a current i.
- Reversing the direction reverses the current.
- Moving the loop induces a current.
- The induced current is set up by an *induced EMF*.

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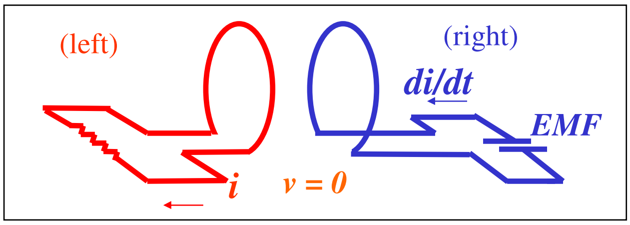
Faraday’s Experiments



- Changing the current in the right-hand coil induces a current in the left-hand coil.
- The induced current does not depend on the size of the current in the right-hand coil.
- The induced current depends on ***di/dt***.

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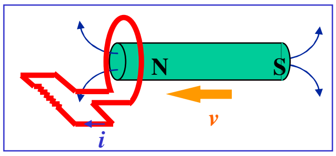
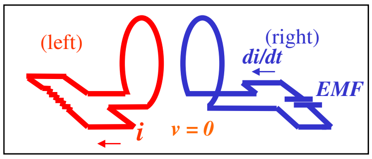
Faraday's Experiments

1) 

2) 

- Moving the magnet changes the flux Ψ (1) – motional EMF.
- Changing the current changes the flux Ψ (2) - transformer EMF.

Faraday: changing the flux induces an EMF (e).

$$e = -\frac{d\Psi}{dt}$$

Faraday's law

The emf induced around a loop

equals the rate of change of the flux through that loop

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Faraday formulated the law named after his name

The induced electromagnetic force (EMF) - e_{emf} or simply (e), in any closed conducting loop (circuit) is equal to the time rate of change of the magnetic flux linkage of the loop.

$$e = -\frac{d\Psi}{dt} = -\frac{d}{dt}\left(\iint_{S_\Gamma} \vec{B} \cdot d\vec{A}\right)$$

Integral form of Faraday's law

The negative sign shows that the induced emf (and currents) would act in such a way that they would oppose the change of the flux creating it. This law is also known as Lentz' law of EMF induction.

If the circuit consists of N loops all of the same area and if Ψ is the flux through one loop, then the total induced *emf* is:

$$e = -\frac{d\Lambda}{dt} = N \cdot \frac{d\Psi}{dt}$$

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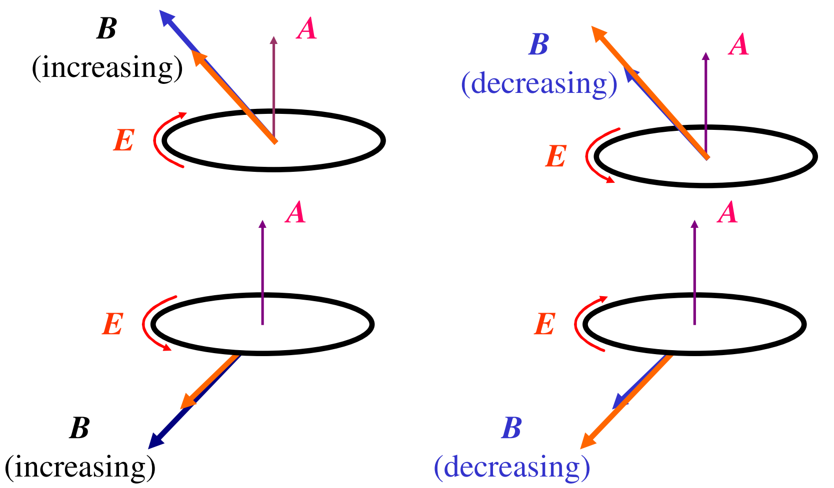
Lenz's Law: The direction of any magnetic induction effect (induced current) is such as to oppose the cause producing it. (Opposing change = inertia!)



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Differential form of Faraday's law

$$e = -\frac{d\Psi}{dt}$$

$$e = \oint_{\Gamma} \vec{E} \cdot d\vec{s} \qquad \Psi = \iint_{S_\Gamma} \vec{B} \cdot d\vec{A}$$

Note: S_Γ is an open surface.

$$\oint_{\Gamma} \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\left(\iint_{S_\Gamma} \vec{B} \cdot d\vec{A} \right)$$

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$$\oint_{\Gamma} \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\left(\iint_{S_\Gamma} \vec{B}\cdot d\vec{A}\right)$$

Applying Stokes's theorem:

$$\oint_{\Gamma} \vec{E}\cdot d\vec{s} = \left(\iint_{S_\Gamma} rot\,\vec{E}\cdot d\vec{A}\right) = -\frac{d}{dt}\left(\iint_{S_\Gamma} \vec{B}\cdot d\vec{A}\right)$$

Suppose that the surface S_Γ is mobile with the velocity *v*, then the derivative with respect the time of the surface integral will be:

$$\frac{d}{dt}\left(\iint_{S_\Gamma} \vec{B}\cdot d\vec{A}\right) = \left(\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}\right) + \iint_{S_\Gamma} rot\left(\vec{B}\times\vec{v}\right)\cdot d\vec{A}$$

The surface is immobile — The surface is mobile with velocity v

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$$\oint_{\Gamma} \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\left(\iint_{S_\Gamma} \vec{B}\cdot d\vec{A}\right) \qquad \frac{d}{dt}\left(\iint_{S_\Gamma} \vec{B}\cdot d\vec{A}\right) = \left(\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}\right) + \iint_{S_\Gamma} rot\left(\vec{B}\times\vec{v}\right)\cdot d\vec{A}$$

$$\Downarrow$$

$$\left(\iint_{S_\Gamma} rot\,\vec{E}\cdot d\vec{A}\right) = -\left(\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}\right) - \iint_{S_\Gamma} rot\left(\vec{B}\times\vec{v}\right)\cdot d\vec{A}$$

$$\Downarrow$$

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} + rot\left(\vec{v}\times\vec{B}\right)$$

Differential form of Faraday's law

Electromagnetic field

The induced electromagnetic force (emf) around a circuit can be separated into two terms:

- Transformer emf component, due to the time-rate of change of the B-field :

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad e_{transformer} = \oint_{\Gamma} \vec{E}\cdot d\vec{s} = \iint_{S_\Gamma} rot\,\vec{E}\cdot d\vec{A} = -\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

- Motional emf component, due to the motion of the circuit:

$$\vec{E} = \vec{v}\times\vec{B} \quad \Rightarrow \quad e_{motional} = \oint_{\Gamma} \vec{E}\cdot d\vec{s} = \oint_{\Gamma} \left(\vec{v}\times\vec{B}\right)\cdot d\vec{s}$$

Note: the induced electric field is non-conservative !!!! $rot\,\vec{E} \neq 0$

The electric field is conservative only in electrostatics regime !!!!

$$rot\,\vec{E} = 0$$

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Electric field in terms of potential functions (electrostatics):

$$rot\,\vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -gradV$$

Electric field in terms of potential functions (electrodynamics): $\vec{B} = rot\,\vec{A}$

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} + rot\left(\vec{v}\times\vec{B}\right) \quad \Rightarrow \quad rot\,\vec{E} = -rot\frac{\partial \vec{A}}{\partial t} + rot\left(\vec{v}\times\vec{B}\right) \quad \Rightarrow \quad rot\left(\vec{E} + \frac{\partial \vec{A}}{\partial t} - \vec{v}\times\vec{B}\right) = 0$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} + \vec{v}\times\vec{B} - gradV$$

The most general form:

Vector magnetic potential

Motional component

Scalar electric potential

In electrodynamics both components are functions of time

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Particular cases:

1) The surface is immobile (v = 0):

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - gradV$$

The motional emf is zero:

$$e_{motional} = \oint_{\Gamma} \vec{E} \cdot d\vec{s} = \oint_{\Gamma} \left(\vec{v} \times \vec{B}\right) \cdot d\vec{s} = 0$$

The transformer emf is non-zero:

$$e_{transformer} = -\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} = \oint_{\Gamma} \vec{E} \cdot d\vec{s} = -\oint_{\Gamma} \frac{\partial \vec{A}}{\partial t} \cdot d\vec{s}$$

Differential form of Faraday's law:

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

2) The surface is mobile but the magnetic field B is constant in time:

$$\vec{E} = \vec{v} \times \vec{B} - gradV$$

The motional emf is non-zero:

$$e_{motional} = \oint_{\Gamma} \vec{E} \cdot d\vec{s} = \oint_{\Gamma} \left(\vec{v} \times \vec{B}\right) \cdot d\vec{s}$$

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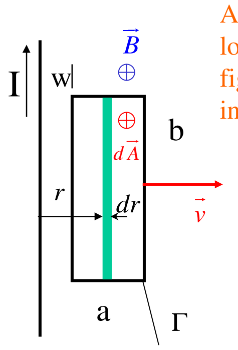
The transformer emf is zero:

$$e_{transformer} = -\iint_{S_\Gamma} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} = 0$$

Differential form of Faraday's law:

$$rot\,\vec{E} = rot\left(\vec{v} \times \vec{B}\right)$$

Applications. Example no.1



A long, straight wire carries a current I (constant). A rectangular loop (a by b) lies at a distance w (at time t = 0), as shown in the figure and it is moved with a constant speed *v*. What is the emf induced in the loop in this case?

$$\Psi = -\frac{d}{dt}\left(\iint_{S_\Gamma} \vec{B} \cdot d\vec{A}\right) = -\frac{d}{dt}\left(\iint_{S_\Gamma} B \cdot dA\right)$$

Because: B and dA are collinear (see the figure)

$$for\ a\ long\ wire: B = \frac{\mu \cdot I}{2 \cdot \pi \cdot r} \qquad and \quad dA = b \cdot dr$$

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$$\Psi=\iint_{S_\Gamma} B\cdot dA=\iint_{S_\Gamma}\frac{\mu\cdot I}{2\cdot\pi\cdot r}\cdot dA=\int_{w+vt}^{w+vt+a}\frac{\mu\cdot I\cdot b}{2\cdot\pi\cdot r}\cdot dr=\frac{\mu\cdot I\cdot b}{2\cdot\pi}\int_{w+vt}^{w+vt+a}\frac{dr}{r}$$

$$\Psi=\frac{\mu\cdot I\cdot b}{2\cdot\pi}\cdot\ln\frac{w+vt+a}{w+vt}$$

$$e=-\frac{d\Psi}{dt}$$

$$e=-\frac{\partial}{\partial t}\Psi=-\frac{\partial}{\partial t}\left(\frac{\mu\cdot I\cdot b}{2\cdot\pi}\cdot\ln\frac{w+vt+a}{w+vt}\right)=$$

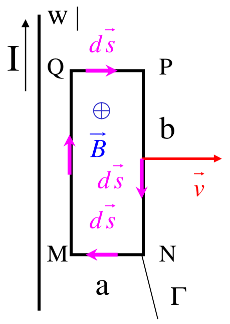
$$=\frac{\mu\cdot I\cdot b}{2\cdot\pi}\left(\frac{v}{w+vt}-\frac{v}{w+vt+a}\right)=\frac{\mu\cdot I\cdot b\cdot v\cdot a}{2\cdot\pi}\cdot\frac{1}{(w+vt+a)\cdot(w+vt)}$$

In this case there is only a motional emf component!!!!

An other way of computation of the motional emf component

$$e_{motional}=\oint_\Gamma \vec{E}\cdot d\vec{s}=\oint_\Gamma(\vec{v}\times\vec{B})\cdot d\vec{s}=0$$

Electromagnetic field



$$e_{motional}=\oint_\Gamma \vec{E}\cdot d\vec{s}=\oint_\Gamma(\vec{v}\times\vec{B})\cdot d\vec{s}=\int_M^N(\vec{v}\times\vec{B})\cdot d\vec{s}+$$

$$+\int_N^P(\vec{v}\times\vec{B})\cdot d\vec{s}+\int_P^Q(\vec{v}\times\vec{B})\cdot d\vec{s}+\int_Q^M(\vec{v}\times\vec{B})\cdot d\vec{s}$$

but $\int_M^N(\vec{v}\times\vec{B})\cdot d\vec{s}=\int_P^Q(\vec{v}\times\vec{B})\cdot d\vec{s}=0$ and

$$\int_P^N(\vec{v}\times\vec{B})\cdot d\vec{s}=-\int_P^N v\cdot B\cdot ds=-v\cdot B\cdot\int_P^N ds=-v\cdot B\cdot b=-v\cdot\frac{\mu\cdot I}{2\cdot\pi\cdot(w+vt+a)}\cdot b$$

$$\int_Q^M(\vec{v}\times\vec{B})\cdot d\vec{s}=\int_Q^M v\cdot B\cdot ds=v\cdot B\cdot\int_Q^M ds=v\cdot B\cdot b=v\cdot\frac{\mu\cdot I}{2\cdot\pi\cdot(w+vt)}\cdot b$$

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$$e_{motional} = \oint_{\Gamma} \left(\vec{v} \times \vec{B}\right) \cdot d\vec{s} = -v \cdot \frac{\mu \cdot I}{2 \cdot \pi \cdot (w + vt + a)} \cdot b + v \cdot \frac{\mu \cdot I}{2 \cdot \pi \cdot (w + vt)} \cdot b$$

$$e_{motional} = \frac{v \cdot \mu \cdot I \cdot b}{2 \cdot \pi}\left(\frac{1}{w+vt} - \frac{1}{w+vt+a}\right) = \frac{v \cdot \mu \cdot I \cdot b \cdot a}{2 \cdot \pi} \cdot \frac{1}{(w+vt)(w+vt+a)}$$

The same as if we applied the integral Faraday's law!!!!!

Example no.2

Solved the same problem using two different methods considering that the current is not any more constant:

$$i = i(t) = I\sqrt{2}\sin(\omega t + \alpha)$$

Electromagnetic field

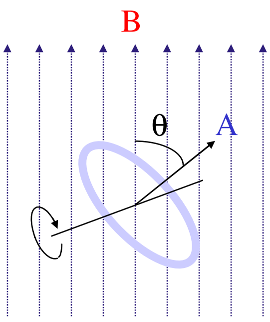
Example nr. 3

Rotating Loop - The Electricity Generator

Consider a loop of area A in a region of space in which there is a uniform magnetic field B. Rotate the loop with an angular frequency ω .

The flux changes because angle θ changes with time: θ = ωt.

Hence:



$$\Psi = \iint_{S_\Gamma} \vec{B} \cdot d\vec{A} = \iint_{S_\Gamma} B \cdot dA \cdot \cos\theta = B \cdot \cos\theta \cdot A$$

$$\cos\theta = \cos(\omega \cdot t + \theta_0)$$

$$\Psi = B \cdot A \cdot \cos(\omega \cdot t + \theta_0)$$

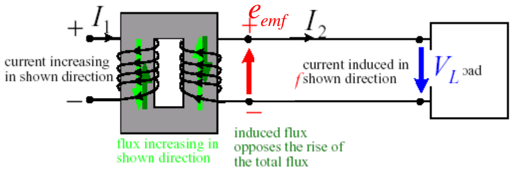
$$e = -\frac{d\Psi}{dt} = B \cdot A \cdot \omega \cdot \sin(\omega \cdot t + \theta_0)$$

This is an **AC (alternating current) generator**

Electromagnetic field

Example nr. 4. Transformer

- A transformer is a device used to increase or decrease an AC voltage. It consists of wire, known as the primary and secondary coils.
- The primary is connected to a source of emf and the secondary to a device usually referred to as the load.



N_S = *the number of secondary turns*

N_P = *the number of primary turns*

$$V_L = -e_{emf} = \frac{\partial \Lambda}{\partial t} = N_S \cdot \frac{\partial \Psi}{\partial t}$$

$$V_L = L_2 \cdot \frac{dI_2}{dt} + M_{21} \cdot \frac{dI_1}{dt}$$

Electromagnetic field

For an ideal transformer:

$$\frac{V_S}{V_P} = \frac{N_S}{N_P}$$

- V_s and V_p are the voltages in the secondary and primary and N_s and N_p are the number of turns in each coil.
- Based on the law of conservation of energy, the power output in the secondary would be equal to the power input in the primary:

The induced emf's react to the total flux change (the self flux change and the mutual flux change), as seen from the equation above. As soon as the total flux levels at zero derivative, the emf's drop to zero. If the total flux starts a decrease, the induced emf's will produce currents whose fluxes will counteract the decrease, i.e. their own fluxes will add up to the dropping total flux.

Electromagnetic field

Maxwell's equations (v = 0)

Integral form	Differential form	Significance
$\oint_{\Gamma} \vec{E} \cdot d\vec{s} = -\frac{\partial \Psi}{\partial t}$	$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Faraday's law **Maxwell 1**
$\oint_{\Gamma} \vec{H} \cdot d\vec{s} = i + \iint_{S} \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A}$	$rot\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	Ampere's law **Maxwell 2**
$\oiint_{\Sigma} \vec{D} \cdot d\vec{A} = Q$	$div\,\vec{D} = \rho_v$	Electric flux's law **Maxwell 3**
$\oiint_{\Sigma} \vec{B} \cdot d\vec{A} = 0$	$div\,\vec{B} = 0$	Magnetic flux's law **Maxwell 4**