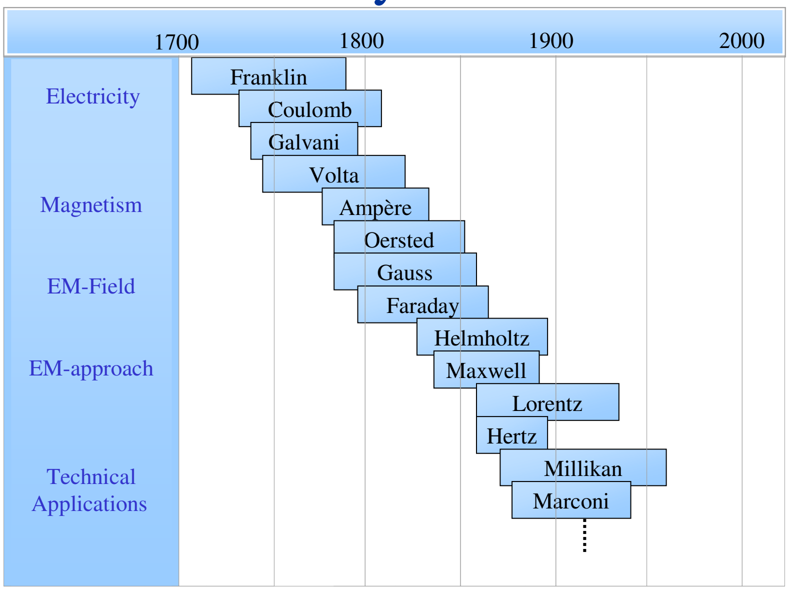
The Theory of Electromagnetic Field

About the Course

- **Aims**
  - To provide students with an introduction to electromagnetism
- **Objectives**
  - To introduce the principles and applications of electromagnetism
  - To understand and apply the laws governing electric and magnetic behaviour
  - To be aware of some of the applications of electromagnetic behaviour

History of EM



INTRODUCTION

Statics and Dynamics in Electromagnetism

Stationary charges → electrostatic fields

(charges have zero velocity and zero acceleration)

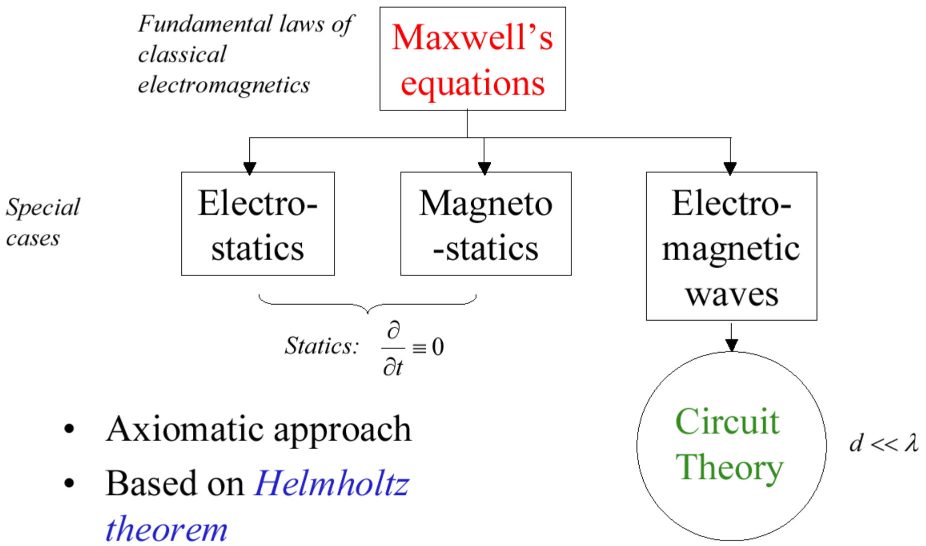
Steady currents → magnetostatic fields

(charges have non- zero velocity and zero acceleration)

Time-varying currents → electromagnetic field

(charges have non-zero velocity and non-zero acceleration)

Approach to Studying Electromagnetics



- Axiomatic approach
- Based on *Helmholtz theorem*

SI (International System of) Units

Fundamental SI Units

Quantity	**Unit**	**Abbreviation**
length	meter	m
mass	kilogram	k
time	second	s
current	ampere	A
temperature	kelvin	K
luminous intensity	candela	cd

Fundamental Vector Field Quantities in Electromagnetics

- Electric field intensity $(\overline{E})$
  units = volts per meter (V/m = kg m/A/s^3)
- Electric flux density (electric displacement) $(\overline{D})$
  units = coulombs per square meter (C/m^2 = A s /m^2)
- Magnetic field intensity $(\overline{H})$
  units = amps per meter (A/m)
- Magnetic flux density $(\overline{B})$
  units = teslas = webers per square meter
  (T = Wb/ m^2 = kg/A/s^3)

Fundamental Vector Field Quantities in Electromagnetics (Cont'd)

- A ***field*** is a spatial distribution of a quantity; in general, it can be either *scalar* or *vector* in nature.
- When an event in one place has an effect on something at a different location, we talk about the events as being connected by a field.
- In general, the fundamental vector field quantities in electromagnetics are vector functions of both position (in three-dimensional space) and time.

Three Universal Constants

- the velocity of an electromagnetic wave (e.g., light) in free space (perfect vacuum)

$$c \approx 3 \times 10^{8} \text{ m/s}$$

- the permeability of free space

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

- the permittivity of free space

$$\varepsilon_0 \approx 8.854 \times 10^{-12} \text{ F/m}$$

Relationships Involving the Three Universal Constants

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

In free space:

$$\overline{B} = \mu_0 \overline{H}$$

$$\boxed{\overline{D} = \varepsilon_0 \cdot \overline{E}}$$

Chapter 1

Electrostatics

Electrostatics

Electric Field in Vacuum

- Basic Definitions
- Electric Charges
- Coulomb's Law and Definition of Electric Force and Intensity
- The Electric Field for discrete distributions
- The Electric Field for Continuous distributions
- Gauss Law

Electrostatics

The concept of a ***field*** is used to describe "***action at a distance***" – a disturbance or input at one point can have an effect or output at a distance point. The region where the effect of this coupling media is felt is the field, described by its (vector) field strength.

Electrical phenomena caused by friction are part of our everyday lives, and can be understood in terms of **electric charge.**

The **effects** of electric charge can be observed in attraction/repulsion of various objects when "charged".

Electrostatics is the branch of electromagnetism dealing with the ***effects of electric charges at rest***.

The fundamental law of **electrostatics is Coulomb's law** which is based on physical observation and cannot be deduced logically or mathematically from any other physical law.

Electrostatics

Electric Charges

The **electric charge** is a fundamental property of matter. It is measured in *Coulombs* (C). It was agreed that the electric current unit *Ampere* (A) would be chosen as a basic unit in SI. Thus, *Coulomb* is a secondary unit derived as:

$$i = -\frac{dQ}{dt} \Rightarrow 1C = 1A \times 1s$$

i is the electric current in *Amperes* (A)

Q is the electric charge in *Coulombs* (C)

t is time

Electrostatics

Electric Charges

- Evidence for electric charges is everywhere, e.g.
  - static electricity
  - lightning
- Objects may become charged by contact and frictional forces.
- Benjamin Franklin (1700's) discovered that there are two types of charges:
  - positive charge
  - negative charge
- Franklin also discovered that **like charges repel** and **unlike charges attract** one another.
- Electric charge is:
  - quantized (Millikan)
  - conserved (Franklin)

Electrostatics

Electric Charges

Classes of Materials

- **Conductors** are materials in which charges may move freely (e.g. copper).
- **Insulators** are materials in which charges cannot move freely (e.g. glass).
- **Semiconductors** are materials in which charges may move under some conditions (e.g. silicon).

Charges and the Earth

- The earth acts as a near-infinite source or sink of charges, and therefore its **net charge** cannot easily be changed.
- Any conductor in contact with the earth is said to be grounded and cannot receive a **net charge**. *(principle of lightning rod)*

Electrostatics

Electric Charges

Induced Charge

- Charged objects brought close to a conductor may cause charge to redistribute *(polarize the conductor).*
- If a polarized conductor is momentarily grounded, charge will be transferred to/from the earth, and it may be left with a net charge (by Induction).
- Objects may be charged by
  - **conduction** *(requires contact with another charged object).*
  - **induction** *(requires no contact with another charged object).*

Electrostatics

Charge is associated with matter. Therefore, it has **finite volume**. However, volume charges Q can be always considered made of even smaller volumes. This is particularly useful when the volume of charges has inhomogeneous charge distribution.

Continuous Distributions of Charges

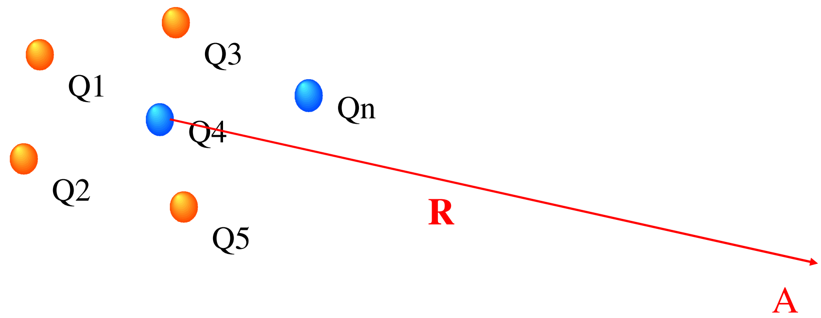
Charges can occurs as:

- **Point charges (C)**
- **Volume charges (C/m^3)** $\Leftarrow$ **most general**
- **Surface charges (C/m^2)**
- **Line charges (C/m)**

Electrostatics

Point charges

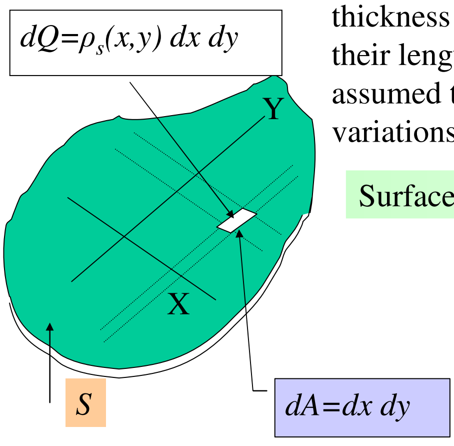
Are **charges** whose volume can be considered infinitesimally small (a point) in comparison with the distance from its center at which its field is analyzed.



Electrostatics

Surface charges

Are useful when the physical 3-D charges are spared in wide thin sheets whose thickness is negligible in comparison with their length and width. Beside, it can be assumed that their charge distribution variations with the height are negligible.



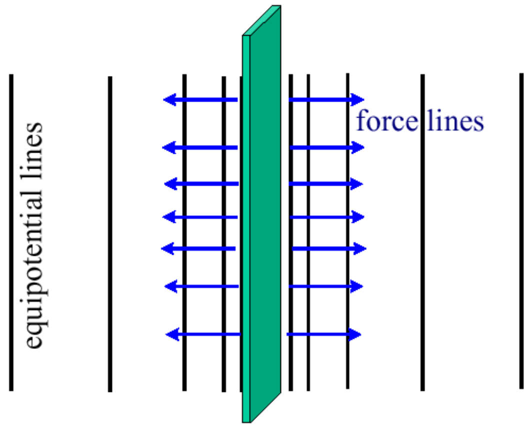
Surface charge density definition [C/m^2]

$$\rho_s = \lim_{\Delta q \to 0} \frac{\Delta Q}{\Delta A} = \frac{dQ}{dA}$$

$$Q = \int_S \rho_s \cdot dA$$

Electrostatics

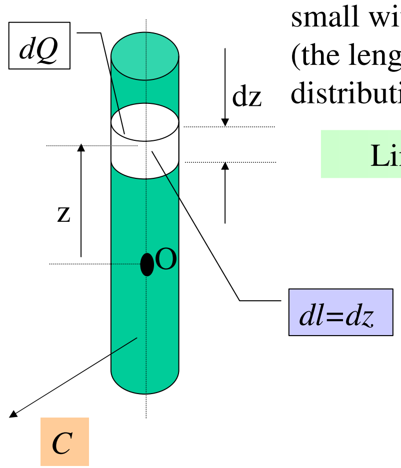
The surface charges imply the symmetry of the field with respect to their plane.



Electrostatics

Line charges

Are useful approximation for charges whose volume has two of its dimensions negligibly small with respect to the other dimensions (the length). The variations of the charge distribution in the cross-section are negligibly.



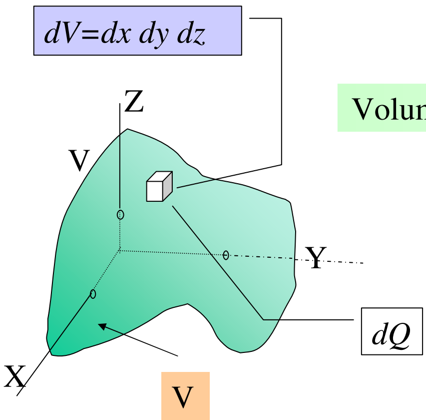
Line charge density definition [C/m]

$$\rho_l = \lim_{\Delta q \to 0} \frac{\Delta Q}{\Delta l} = \frac{dQ}{dl}$$

$$Q = \int_C \rho_l \cdot ds$$

Electrostatics

Volume charges



Volume charge density definition [C/m³]

$$\rho_v = \lim_{\Delta q \to 0} \frac{\Delta Q}{\Delta v} = \frac{dQ}{dv}$$

$$Q = \int_V \rho_v \cdot dv$$

Electrostatics

2. Coulomb's law (1785)

$$F_{12} = \underbrace{\frac{1}{4\pi\varepsilon}}_{k} \frac{Q_1 Q_2}{R^2} = -F_{21}, \quad \text{N}$$

Coulomb's law states that the force between two electrical charges at rest is proportional to the amount of their charge and inverse proportional to the square of the distance between them.

The inverse square law is a universal property of fields in our 3-D world.

Charges of the same sign repel each other, and charges of the opposite signs attract each other.

Electrostatics

The constant of proportionality k depends on the system of units used. In SI $k = 1/(4\pi\varepsilon)$

$$[k] = \frac{\mathrm{N}\times\mathrm{m}^2}{\mathrm{C}^2} = \frac{\mathrm{N}\times\mathrm{m}^2}{\mathrm{A}^2\times\mathrm{s}^2} = \frac{\mathrm{V}\times\mathrm{m}}{\mathrm{C}}$$

By experiment (in air/vacuum), if the force is measured in newtons, the distance in meters, and the charge in ampere-seconds (coulombs):

$$k = 9.0\times10^9$$

Theoretically, this constant in the SI system must be exactly

$$k = 10^{-7}c^2$$

where c is the speed of light.

Electrostatics

The constant

$$\varepsilon = \frac{1}{4\pi k}$$

is called *dielectric permittivity*, which in vacuum is

$$\varepsilon_0 = \frac{1}{4\pi c^2\cdot10^{-7}} \simeq \frac{1}{4\pi\cdot9\cdot10^9} = \frac{10^{-9}}{36\pi}$$

A more precise value is

$$\varepsilon_0 = 8.856\times10^{-12},\quad \mathrm{F/m{=}C/(V\times m)}$$

The dielectric permittivity of materials depends on the ability of matter to polarize under the influence of external electric field.

Electrostatics

The dielectric permittivity of matter is usually specified relative to that of vacuum via the relative dielectric permittivity (dielectric constant) ε_r

$$\varepsilon = \varepsilon_r \varepsilon_0$$

For air: $\varepsilon_r = 1.0006$

For water: $\varepsilon_r = 80$

Urban (dry) ground: $\varepsilon_r \simeq 3$

Rural (moist) ground: $\varepsilon_r \simeq 14$

Electrostatics

The electric field (intensity) vector $\vec{E}$

The electric field vector is the force exerted on a unit force.

$$\vec{E} = \lim_{\Delta q \to 0} \frac{\Delta \vec{F}}{\Delta q} = \frac{d\vec{F}}{\Delta q}, N/C = V/m \quad \Leftrightarrow \vec{F} = \Delta q \cdot \vec{E}, \ N$$

Here, Δq is a test (probe) charge, which means that it is small enough not to disturb the measured original field of the source charge ***Q*****.**

Electrostatics

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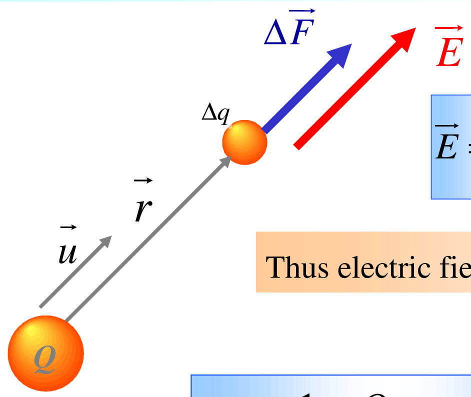
Electrostatics

Electric field intensity of a single point charge q



$$\vec{E} = \lim_{\Delta q \to 0} \frac{\Delta \vec{F}}{\Delta q}$$

Thus electric field from a single charge is:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \vec{u} = \frac{Q}{4\pi\varepsilon_0} \frac{\vec{r}}{r^3}$$

Electrostatics

Electric field intensity

General Properties of an Electric Field

- Electric field is generated by any charged object.
- It is a vector field and obeys the **principle of superposition,** i.e., the field of a system of charged objects is equal to the (vector) sum of the field of each individual charged object in the system.
- The electrostatic force between charged objects is mediated by the electric field.

Electrostatics

Electric field intensity

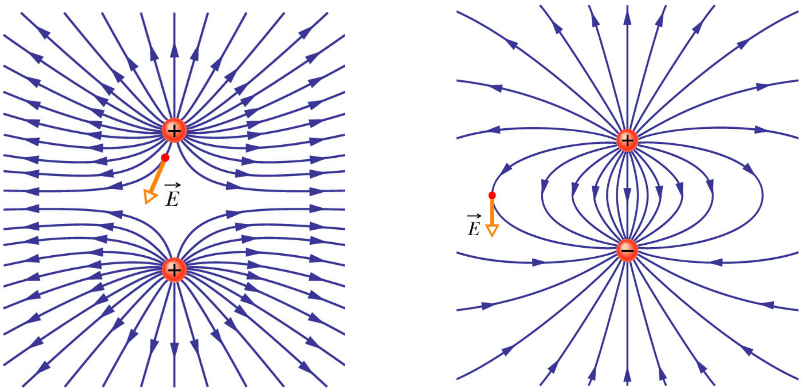
Electric Field Lines

- A visualization tool to illustrate the geometry of an electric field.
- Electric field lines originate from positive charges and terminates at negative charges.
- The direction of the electric field at any location is tangential to the field line there.
- The magnitude of the electric field at any location is proportional to the density of the lines there.

Electrostatics

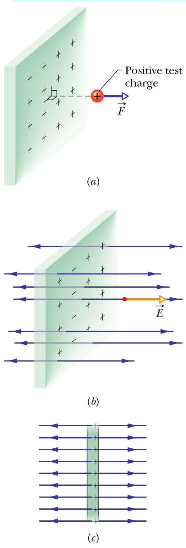
Electric field intensity

Examples:



Electrostatics

Electric field intensity



Examples:

Uniform electric field: an electric field that has the same magnitude and direction at every point.

Electrostatics

Electric Field due to Multiple Point Charges

The force on a test charge is then given by

$$\Delta\vec{F} = \sum_{i=1}^{n} \Delta\vec{F}_i$$

so the electric field is, by definition, given by

$$\vec{E} = \lim_{\Delta q \to 0} \frac{\Delta\vec{F}}{\Delta q} = \lim_{\Delta q \to 0} \frac{\sum_{i=1}^{n} \Delta\vec{F}_i}{\Delta q} = \sum_{i=1}^{n} \vec{E}_i$$

Principle of superposition!

Electrostatics

Electric Field due to Multiple Point Charges

Principle of superposition

This principle is of major importance to the solution of field problems in linear media, i.e. media whose electromagnetic (EM) properties do not depend on the field's intensity. In the case of electrostatic fields (ESF), the EM property that matters is the dielectric permittivity. If it does not depend on E, then the medium is linear.

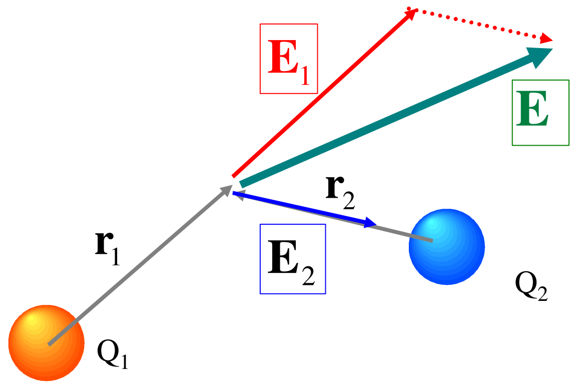
The ESF of multiple charges at any point is a vectorial sum of the fields created by each individual charge:

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i$$

Electrostatics

Electric Field due to Multiple Point Charges

Principle of superposition (cont)



Electrostatics

Electric field due to continuous charges distributions

When the field is due to charges, which are distributed throughout a **volume** with some known density ρ_v C/m^3, then this volume is represented by an **infinite number of infinitesimal** (differential) **charges**:

$$dQ = \rho_v \cdot dv, \quad C$$

Each differential charge is in **effect a point charge**. Thus, it generates a **differential "part" of the field**, which is:

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2}\vec{u}, \quad V/m$$

Electrostatics

Electric field due to continuous charges distributions

The **total field** is obtained via the **principle of superposition**.

A summation over differential contributions has to be performed: this is integration (volume integration)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \int_V \frac{\rho_v}{r^2} \frac{\vec{r}}{r} dv, \quad V/m$$

When **distributed surface** charge is present, it is broken down into differential surface charges, each of which is described by its surface density ρ_s C/m^2:

$$dQ = \rho_s \cdot dA, \quad C$$

Electrostatics

Electric field due to continuous charges distributions

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \int_S \frac{\rho_s}{r^2} \frac{\vec{r}}{r} dA, \quad V/m$$

When distributed line charge is present, it is broken down into differential line charges, each of which is described by its line density ρ_l C/m:

$$dQ = \rho_l \cdot ds, \quad C$$

Thus, the field generated by linear charges is found by the following line integrals:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \int_C \frac{\rho_l}{r^2} \frac{\vec{r}}{r} ds, \quad V/m$$

Electrostatics

Electric field due to continuous charges distributions

Electric field computation using the superposition principle

Algorithm of computation:

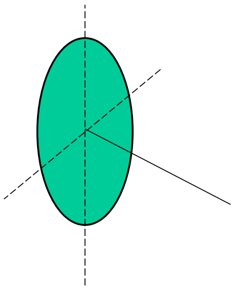
- Analysis and symmetry
- Approach to solution
- Calculations
- Conclusions

Electrostatics

Electric field due to continuous charges distributions

Example no.1

Electric field of a thin disk



Available :

A thin circular disk with radius *R* and charge density ρ [C/m^2]

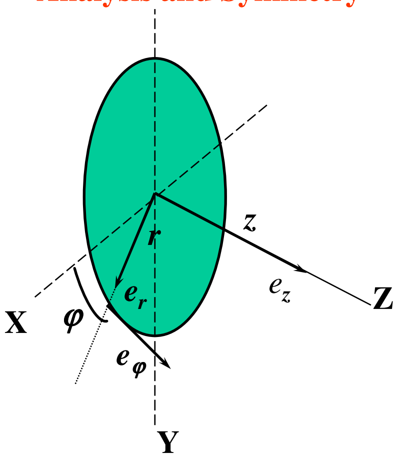
Question :

Calculate ***E***-field in arbitrary points a both sides of the disk

Electrostatics

Electric field due to continuous charges distributions

Analysis and Symmetry



1. Charge distribution:

   ρ_s [C/m^2]

2. Coordinate axes:

   Z-axis = symm. axis,

   perpend. to disk

3. Symmetry circle: cylinder

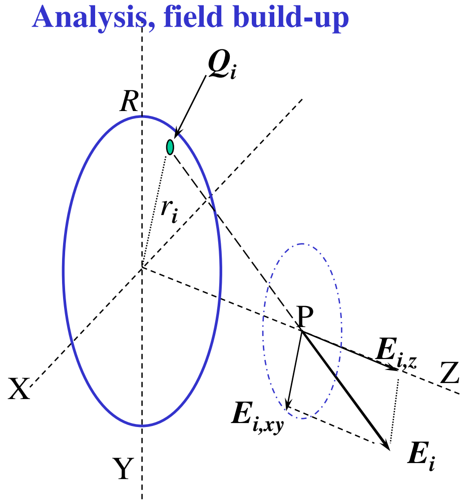
4. Cylinder coordinates:

   r, z, φ

Electrostatics

Electric field due to continuous charges distributions

Analysis, field build-up



1. XYZ-axes
2. Point P on Y-axis
3. all Q_i's at r_i *and* φ_i contribute $\boldsymbol{E}_i$ to $\boldsymbol{E}$ in P
4. $\boldsymbol{E}_{i,xy}$, $\boldsymbol{E}_{i,z}$
5. expect: $\Sigma\, \boldsymbol{E}_{i,xy} = 0$,

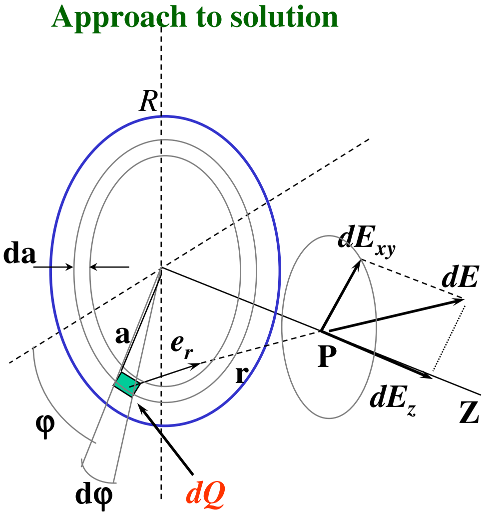
   to be checked !!

6. $\boldsymbol{E} = E_z \boldsymbol{e}_z$ ***only !***

Electrostatics

Electric field due to continuous charges distributions

Approach to solution

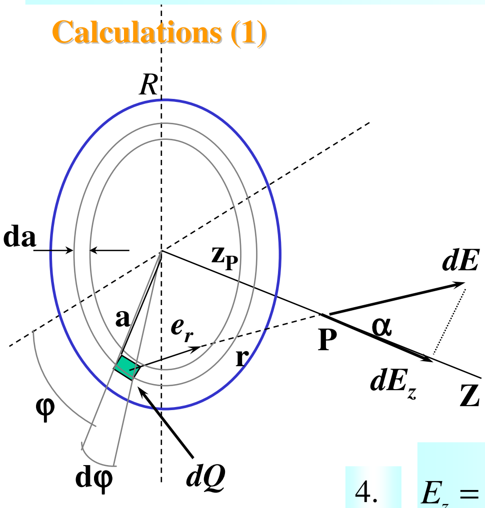


1. Distributed charges
2. $dE = \frac{dQ}{4\pi\varepsilon_0 r^2} e_r$
3. Rings and segments
4. $dQ = \rho_s.dA = \rho_s\ (da.)(a\ d\varphi)$
5. z- component only !
6. $dE_z = \frac{dQ}{4\pi\varepsilon_0 r^2}(e_r \bullet e_z)$

Electrostatics

Electric field due to continuous charges distributions

Calculations (1)

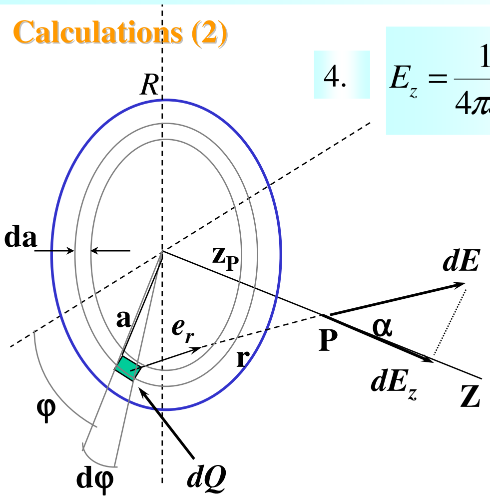


1. $dE_z = \frac{dQ}{4\pi\varepsilon_0 r^2}(e_r \bullet e_z)$
2. $dQ = \rho_s.dA = \rho_s\ (da.)(a\ d\varphi)$
3. $e_r \bullet e_z = \cos\alpha = \frac{z_P}{\sqrt{a^2 + z_P{}^2}}$
4. $E_z = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{\rho_s.da.a.d\varphi}{\left(a^2 + z_P{}^2\right)} \frac{z_P}{\sqrt{a^2 + z_P{}^2}}$

Electrostatics

Electric field due to continuous charges distributions

Calculations (2)



4. $E_z = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{\rho_s.da.a.d\varphi}{\left(a^2+z_P{}^2\right)}\frac{z_P}{\sqrt{a^2+z_P{}^2}}$

5. $E_z = \frac{\rho_s}{2\varepsilon_0}\left[1-\frac{z_P}{\sqrt{z_P{}^2+R^2}}\right]$

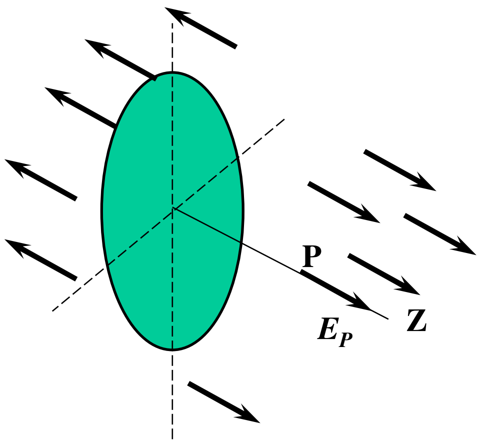
6. If R –> *infinity :*

$$E_z = \frac{\rho_s}{2\varepsilon_0}$$

Electrostatics

Electric field due to continuous charges distributions

Conclusions



for infinite disk:

$$\mathbf{E}_P = \frac{\rho_s}{2\varepsilon_0}\mathbf{e_z}$$

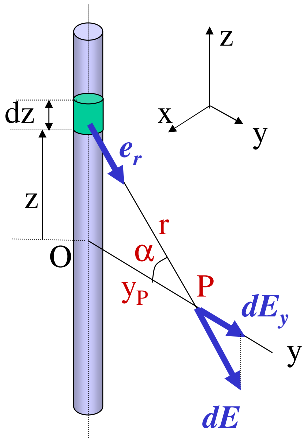
field strength independent of distance to disk =>

homogeneous field

Electrostatics

Analysis:

- ∞ long wire: ρ_l [C/m]
- cylindrical symmetry



Problem: E_P in point P(0,y_P,0)

Approach: $$dE = \frac{dQ}{4\pi\varepsilon_0 r^2} e_r$$

- Charge element: $dQ = \rho_l\, dz$
- Symmetry ⇒ y-component only !!

$e_r \cdot e_y = \cos\alpha$

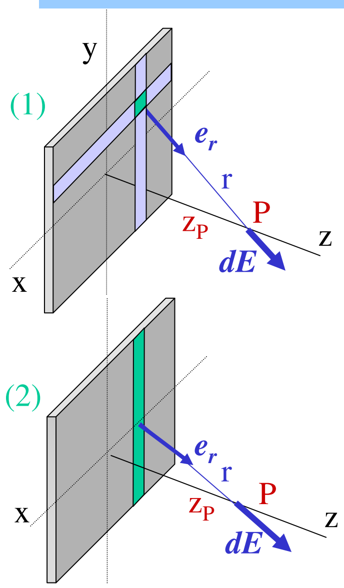
$$\mathbf{E} = \int d\mathbf{E}_y = \int_{-\infty}^{\infty} \frac{\rho_l . dz}{4\pi\varepsilon_0 r^2} \cos\alpha \; = \frac{\rho_l}{2\pi\varepsilon_0 y_P}$$

r, e_r and α are f(z) :

$$r = \sqrt{y_P{}^2 + z^2} \; ; \cos\alpha = \frac{y_P}{r}$$

Conclusion: ***E*** radial symmetry

Electrostatics



Thin plate , ρ_s [C/m²]

(1) ▱ = dA = dx.dy , at (x,y)

$dQ = \rho_s\, dA = \rho_s\, dx.dy$

$\boldsymbol{dE} = dE_x\, \boldsymbol{e_x} + dE_y\, \boldsymbol{e_y} + dE_z\, \boldsymbol{e_z}$

if plate ∞ large : $\boldsymbol{dE} \,//\, \boldsymbol{e_z}$

(2) if $\rho_s = f(x)$ only:

Use result for ∞ long wire:

$$dE_z = \frac{d\rho_l}{2\pi\varepsilon_0 z_P}$$ ***dE*** in XZ-plane

with $d\rho_l = \rho_s . dx$

Electrostatics

Summary of field formulas for standard charge distributions

Point charge: $\vec{E} = \frac{1}{4\pi\varepsilon}\frac{Q}{R^2}\hat{R}, \quad \text{V/m}$

Infinite uniform line charge:

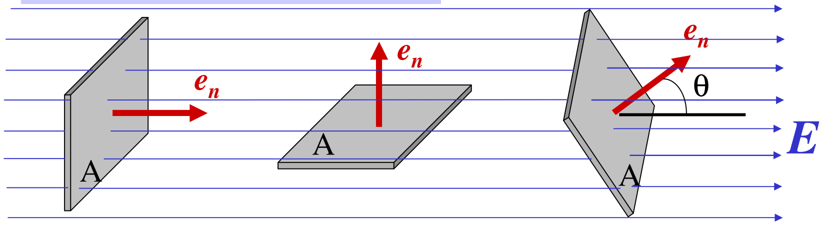
$$\vec{E} = \frac{1}{2\pi\varepsilon}\frac{\rho_l}{r}, \quad \text{V/m}$$

Infinite uniform surface charge:

$$\vec{E} = \hat{n}\frac{\rho_s}{2\varepsilon}, \quad \text{V/m}$$

Electric Flux Φ : definition

1. Homogeneous vector field **E**



θ ≡ < (e_n, ***E***) = 0

Def.: Φ = c.A.E
Choice: c ≡ 1

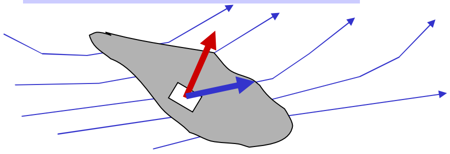
θ = 90°

Φ = 0

Φ = E A cos θ

Φ = (***E***.e_n) A

2. General vector field ***E***

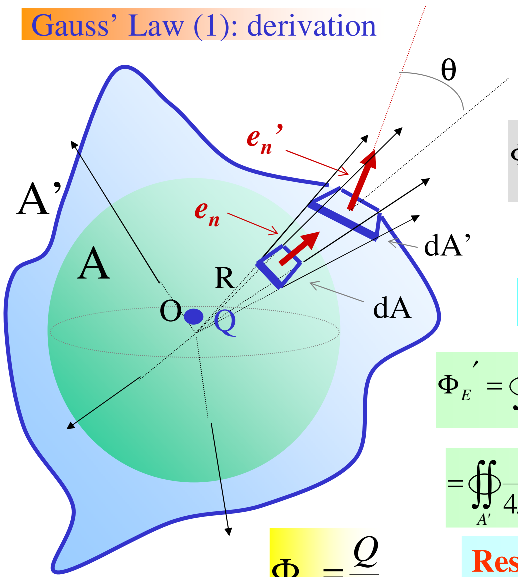


For small surface elements dA:
A and e_n are constant

$$\Phi = \iint_S \vec{E}\cdot\mathbf{e}_n dA = \iint_S \vec{E}\cdot\mathbf{d}\vec{A}$$

Electrostatics

Gauss' Law (1): derivation



Charge Q in O

Flux Φ_E through sphere A:

$$\Phi_E = \oiint_A \mathbf{E}\cdot d\mathbf{A} = \oiint_A \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{e_r}\cdot\mathbf{e_n}\,dA$$

$$= \frac{Q}{4\pi\varepsilon_0 R^2}.1.4\pi R^2 = \frac{Q}{\varepsilon_0}$$

Flux Φ_E' through surface A':

$$\Phi_E' = \oiint_{A'} \mathbf{E}'\cdot d\mathbf{A}' = \oiint_{A'} \frac{Q}{4\pi\varepsilon_0 r'^2}\mathbf{e_{r'}}\cdot\mathbf{e_n}'\,dA'$$

$$= \oiint_{A'} \frac{Q}{4\pi\varepsilon_0}\frac{\cos\theta.dA'}{r'^2} = \oiint_{A'} \frac{Q}{4\pi\varepsilon_0}\frac{dA}{R^2} = \frac{Q}{\varepsilon_0}$$

$$\Phi_E = \frac{Q}{\varepsilon_0}$$

Result is independent of the shape of the surface !!!

Electrostatics

Gauss' Law (1): continuation

Flux: $\Phi_E = \dfrac{Q}{\varepsilon_0}$

The total flux through a closed surface is equal to the total charge enclosed divided by the vacuum permittivity.

Result is independent of the shape of the surface

Consequences:

- Q needs not to be in O
- charge outside: no net flux
- more charges in A:

$$\Phi_E = \sum_i \Phi_{E,i}$$

Electrostatics

How do we use the Gauss' theorem for:

- **Find the E-field**
  - if we know the charge distribution and
  - it has enough symmetry to let us evaluate the integral.
- **Find the charge distribution:**
  - charges on conducting spheres,
  - charge within a given volume.

Electrostatics

Calculating the E-field from Gauss's Law

- First **find a surface**, the Gaussian surface to whose area vector is parallel to the E-field;

$$\oint \vec{E} d\vec{A} = \oint E dA$$

- Also ensure that on the surface the magnitude of of the **E-field is constant.**

$$\oint E dA = E \oint dA = EA$$

- Use geometry to evaluate A.
- Gauss's law then relates E to the charge inside the surface.

Electrostatics

Calculating the E-field from Gauss's Law

More Generally

- Use the **symmetry** of the charge distribution to determine the pattern of the field lines.
- Choose a **Gaussian surface** so that E is parallel to A, or can sensibly be divided into parallel and perpendicular components, since:

$$\vec{E}_{perp}.\vec{A} = 0$$

- If E is parallel to A, make sure that E is constant over the area.

Electrostatics

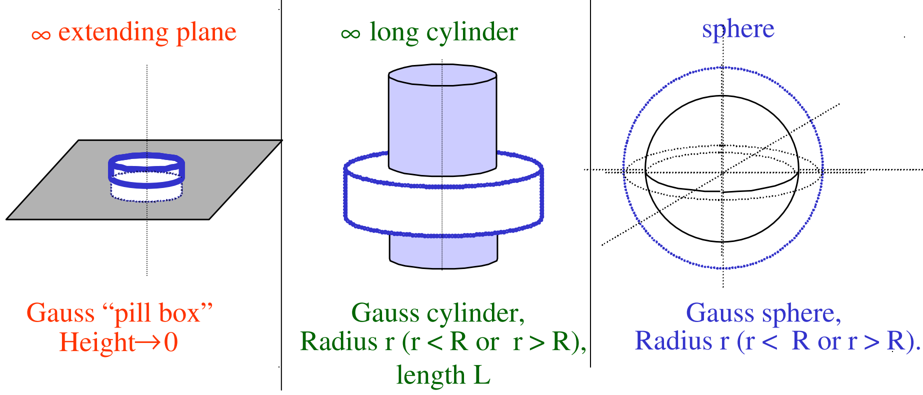
Electric field computation using the Gauss Law

Algorithm of computation:

- Analysis and symmetry
- Approach to solution
- Calculations
- Conclusions

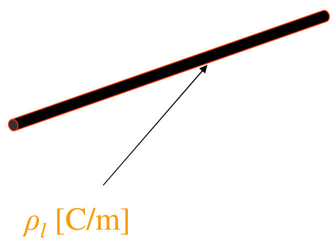
Electrostatics

Basic symmetries for Gauss' Law



Electrostatics

Gauss' Law for a long straight line of charge (linear distribution)



Available:

Infinitely long straight line, carrying charge density ρ_l [C/m]

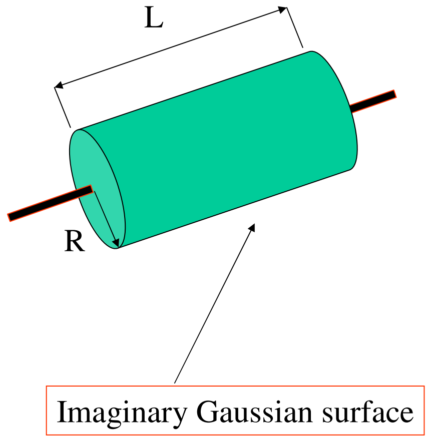
Question:

Calculate ***E***-field in arbitrary points outside of the line

Electrostatics

Gauss' Law for a long straight line of charge (linear distribution)

Analysis and Symmetry

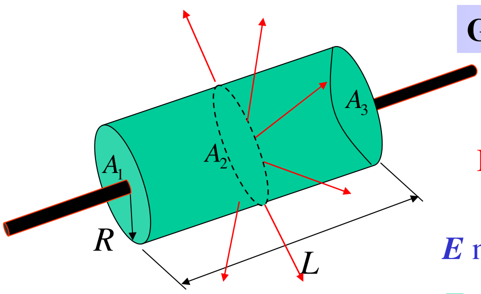


1. Infinitely long line.
2. Charge distribution: homogeneous. ρ_l [C/m].
3. Cylinder symmetry.

Consequences:
The Gauss imaginary box:
a coaxial cylinder of radius R,
length L.

Electrostatics

Approach to solution and computation



Gauss' Law: $\iint_{\Sigma} \vec{E} \bullet \vec{dA} = \frac{Q}{\varepsilon_0}$

Due to the symmetry on the Σ Gauss box:

E normal and constant to surface A_2

E parallel to surfaces A_1 and A_3

$$\Phi_E = \int_{\Sigma} \vec{E} \bullet d\vec{A} = \underbrace{\int_{A_1} \vec{E} \bullet d\vec{A}}_{\text{Front}} + \underbrace{\int_{A_3} \vec{E} \bullet d\vec{A}}_{\text{Back}} + \underbrace{\int_{A_2} \vec{E} \bullet d\vec{A}}_{\text{Side}}$$

Electrostatics

Computation

$$\Phi_E = \int_{\Sigma} \vec{E} \bullet d\vec{A} = \int_{A_1} \vec{E} \bullet d\vec{A} + \int_{A_2} \vec{E} \bullet d\vec{A} + \int_{A_3} \vec{E} \bullet d\vec{A}$$

$$\vec{E} \bullet d\vec{A} = 0 \qquad \int_{A_2} \vec{E} \bullet d\vec{A} = \int_{A_2} E \cdot dA \qquad \vec{E} \bullet d\vec{A} = 0$$

Due to the symmetry E constant on A_2

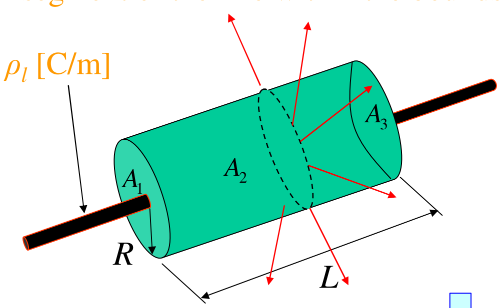
$$\Phi_E = \int_{A_2} E \cdot dA = E \cdot \int_{A_2} dA = E \cdot A_2 = E \cdot \underbrace{2 \cdot \pi \cdot R \cdot L}_{\text{Lateral area of cylinder}}$$

Lateral area of cylinder

Electrostatics

Computation

The total enclosed charge by the Gaussian surface is only the segment of the line within the bounds of the front and back surfaces.



$$Q(enclosed) = \rho_\ell \cdot L$$

$$\Phi_E = \iint_{\Sigma} \vec{E} \bullet d\vec{A} = \frac{Q}{\varepsilon_0}$$

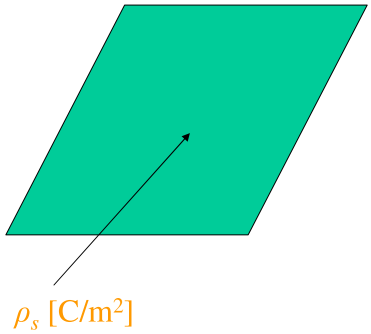
$$\Phi_E = E \cdot 2 \cdot \pi \cdot R \cdot L$$

$$\vec{E} = \frac{\rho_l}{2 \cdot \pi \cdot \varepsilon_0 \cdot R} \cdot \vec{u} = \frac{\rho_l}{2 \cdot \pi \cdot \varepsilon_0 \cdot R} \cdot \frac{\vec{R}}{R}$$

Important !!

Electrostatics

Gauss' Law for an infinite flat surface with charge distribution



<u>Available:</u>

Infinitely flat charge distribution, carrying charge density ρ_s [C/m^2]

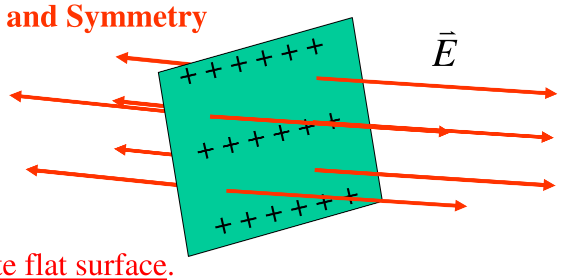
<u>Question:</u>

Calculate ***E***-field in arbitrary points outside of the infinite surface

Electrostatics

Gauss' Law for an infinite flat surface with charge distribution

Analysis and Symmetry



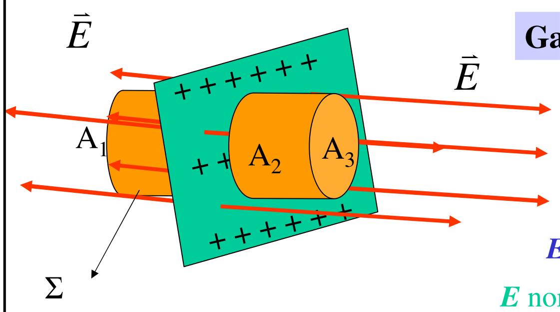
1. <u>Infinite flat surface</u>.
2. <u>Charge distribution:</u> homogeneous. ρ_s [C/m^2].
3. <u>Plan parallel symmetry.</u>
   The electric field is uniform and normal to the flat surface.

Consequences:

The Gauss imaginary box a cylindrical surface with the top and bottom surfaces parallel to the charged plane

Electrostatics

Approach to solution and computation



Gauss' Law: $\iint_\Sigma \vec{E}\bullet\vec{dA} = \frac{Q}{\varepsilon_0}$

Due to the symmetry on the Σ Gauss box:

E parallel to surface A_2

E normal and constant to surfaces A_1 and A_3

$$\Phi_E = \int_\Sigma \vec{E}\bullet d\vec{A} = \underbrace{\int_{A_1} \vec{E}\bullet d\vec{A}}_{\text{Front}} + \underbrace{\int_{A_3} \vec{E}\bullet d\vec{A}}_{\text{Back}} + \underbrace{\int_{A_2} \vec{E}\bullet d\vec{A}}_{\text{Side}}$$

Electrostatics

Computation

$$\Phi_E = \int_\Sigma \vec{E}\bullet d\vec{A} = \int_{A_1} \vec{E}\bullet d\vec{A} + \int_{A_3} \vec{E}\bullet d\vec{A} + \int_{A_2} \vec{E}\bullet d\vec{A}$$

$$\int_{A_1} \vec{E}\bullet d\vec{A} = \int_{A_1} E\cdot dA \qquad \int_{A_3} \vec{E}\bullet d\vec{A} = \int_{A_3} E\cdot dA \qquad \vec{E}\bullet d\vec{A} = 0$$

Due to the symmetry E constant on A_1 and A_3

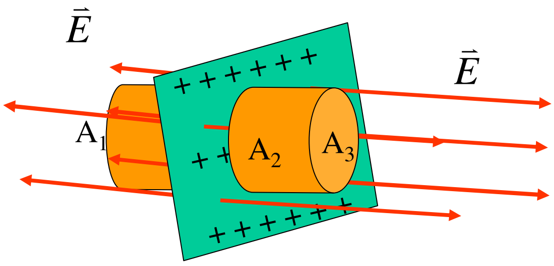
$$\Phi_E = \int_{A_1} E\cdot dA + \int_{A_3} E\cdot dA = E\cdot\int_{A_1} dA + E\cdot\int_{A_3} dA = 2\cdot \underbrace{E\cdot A}$$

Top or bottom area of cylinder

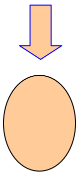
Electrostatics

Computation

The total enclosed charge by the Gaussian surface is only the charge on the disk contained within the cylinder.



Area $A=A_1=A_3$



$$Q(enclosed) = \rho_s \cdot A$$

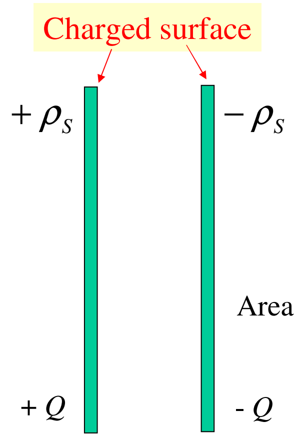
$$\Phi_E = 2 \cdot E \cdot A$$

$$\vec{E} = \frac{\rho_s}{2 \cdot \varepsilon_0} \cdot \vec{u}$$

Uniform field Important !!

Electrostatics

Electric Field for uniformly charged parallel plate capacitor



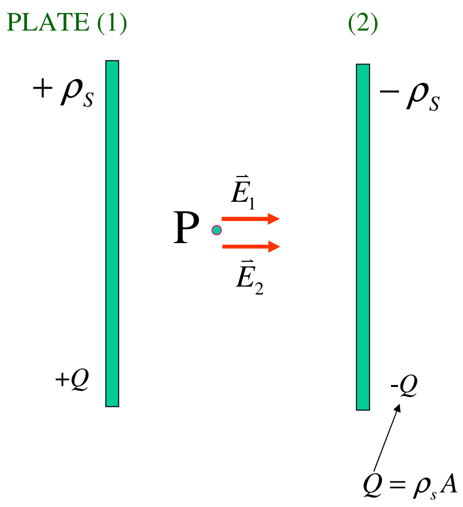
In this example we will make use of the result obtained from the infinite charged flat surface in order to determine the electric field inside and outside a parallel plate capacitor.

The plane of the plates are taken to extend to infinity. This is usually a good approximation when the plate separation is small compared to their area.

Area of plates A

Electrostatics

Electric Field for uniformly charged parallel plate capacitor



For a point P between plates

By superposition:

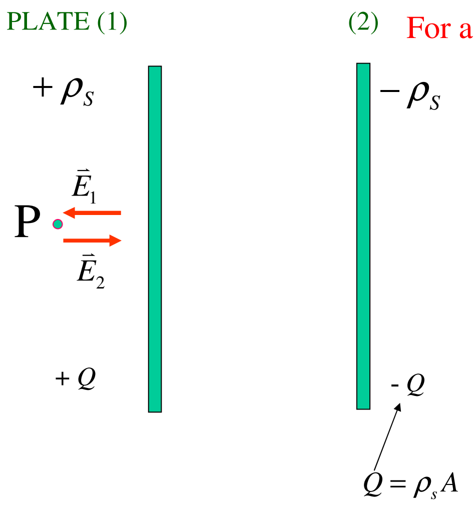
$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$\vec{E} = \left(\frac{Q}{2\varepsilon_o A} + \frac{Q}{2\varepsilon_o A} \right) \cdot \vec{u}$$

$$\boxed{\vec{E} = \frac{Q}{\varepsilon_o A} \cdot \vec{u}} \Rightarrow \boxed{\vec{E} = \frac{\rho_s}{\varepsilon_o} \cdot \vec{u}}$$

Electrostatics

Electric Field for uniformly charged parallel plate capacitor



For a point P to the left of both plates

By superposition:

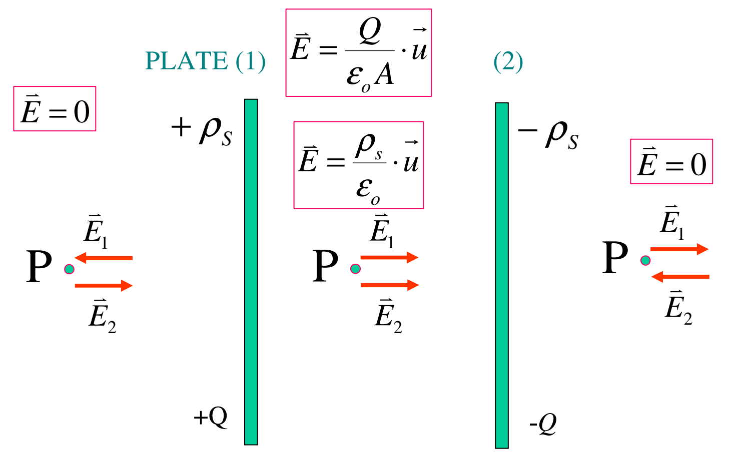
$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$\vec{E} = \left(-\frac{Q}{2\varepsilon_o A} + \frac{Q}{2\varepsilon_o A} \right) \cdot \vec{u}$$

$$\boxed{\vec{E} = 0}$$ On Left side

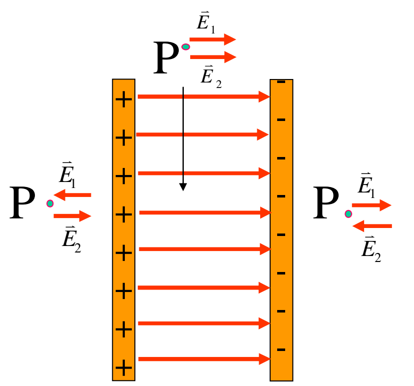
Electrostatics

Electric Field for uniformly charged parallel plate capacitor

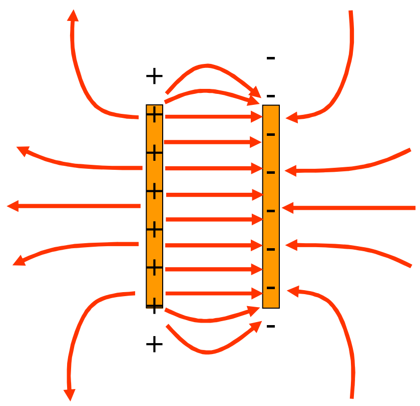


Electrostatics

Electric Field for uniformly charged parallel plate capacitor



Ideal infinite parallel plate capacitor



Real finite parallel plate capacitor

Electrostatics

Electric Potential

- The electric potential definition
- Equipotential surfaces
- Potential due to a continuous distribution of charges
- Calculating the electric field from the potential

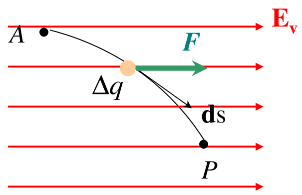
Electrostatics

Electric Potential

When a test charge Δq moves from A to P *(named origin of potential)* in a region of electric field **E** the field does work on the charge. For an infinitesimal displacement d**s** the work done by the field is:

$$L_{AP} = \int_A^P \vec{F} \cdot d\vec{s} = \int_A^P \Delta q \cdot \vec{E}_v \cdot d\vec{s} = \Delta q \int_A^P \vec{E}_v \cdot d\vec{s}$$

The **electric potential** is defined as the potential energy *per unit charge*, and is independent of the test charge Δq.It has a unique value at every point in an electric field.



$$V_A = \frac{L_{AP}}{\Delta q} = \int_A^P \vec{E}_v \cdot d\vec{s}$$

Electrostatics

Electric Potential

For V_A to be meaningful we must define where the electric potential is zero in P. This can be chosen arbitrarily, and common choices are the earth (ground) or at an infinite distance from the sources of the electric field (only valid if the source is located on a finite domain).

Properties of the electric potential

- electric potential is a scalar field.
- electric potential does not depend on the chosen path of integration.
- electrostatic field is a conservative field !!!!
- SI units of electric potential:

$$V_A = \frac{L_{AP}}{\Delta q}$$ → Units are J/C → **Volts (V)**

Electrostatics

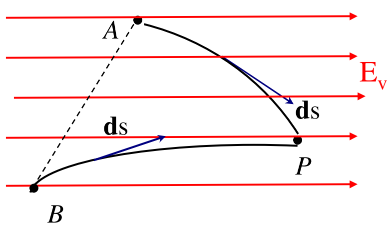
The electric potential difference

The potential difference $U_{AB} = V_B - V_A$ between two points A and B is then:

$$U_{AB} = V_A - V_B = \int_A^P \vec{E}_v \cdot d\,\vec{s} - \int_B^P \vec{E}_v \cdot d\,\vec{s} = \int_A^P \vec{E}_v \cdot d\,\vec{s} + \int_P^B \vec{E}_v \cdot d\,\vec{s} = \int_A^B \vec{E}_v \cdot d\,\vec{s}$$

Since the electric force is conservative, the **electric potential difference** does not depend on the integration path, but on the initial and final points. Do not dependent on the origin of potential.

If we considered a closed curve then:



$$U_\Gamma = \oint_\Gamma \vec{E}_v \cdot d\,\vec{s} = 0$$

New units for electric field from
$E = (V_A - V_B)/s$ uniform field, so **E** has units of V/m
Note: 1 N/C = 1 V/m

Electrostatics

Equipotential surfaces and equipotential lines

Recall the definition of the potential difference between two points A and B:

$$V_A - V_B = \int_A^B \vec{E}_v \cdot \overrightarrow{ds}$$

Equipotential points: points where $V_A = V_B$.

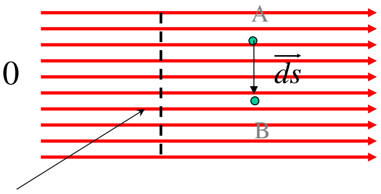
$$\int_A^B \vec{E}_v \cdot \overrightarrow{ds} = 0 \quad \Rightarrow \quad \vec{E}_v \cdot \overrightarrow{ds} = 0$$

Any surface, planar or curved, over which the potential is constant is called an equipotential surface. The equipotential surface may or may not coincide with a physical surface.

Electrostatics

Equipotential surfaces and equipotential lines

$$\vec{E}_v \cdot d\vec{s} = 0$$



$\vec{E}_v$

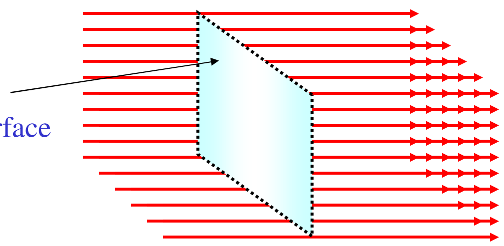
$\left.\begin{array}{l}\vec{E}_v \\ d\vec{s}\end{array}\right\}$ perpendicular

Equipotential lines

Collection of all points at same potential form a line in a 2-D view.

Equipotential surface

Collection of all points at same potential forms a surface in a 3-D view.



$\vec{E}_v$

Electrostatics

Equipotential lines equation:

$$\vec{E}_v \cdot \vec{ds} = 0$$

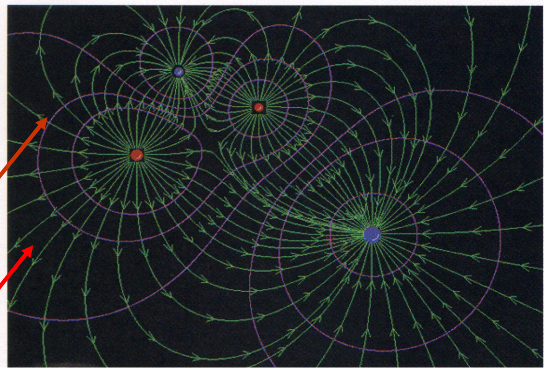
Electric field lines equation:

$$\vec{E}_v \times \vec{ds} = 0$$

Equipotential lines are orthogonal to the electric field lines !!!

Example

Electric field of various charges



Equipotential, E field

Electrostatics

Potential for point charges

For a collection of point charges the electric potential is found using the superposition principle:

$$V = \sum_i V_i$$

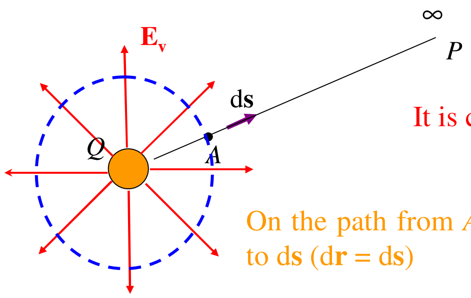
Since V is a scalar it is much easier to evaluate than the vector electric field $\mathbf{E}_v$.

Unsolved question:

If we know V how it is possible to find E?

Electrostatics

Electric potential for a single point charges



$$\vec{E}_v = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^2}\cdot\frac{\vec{r}}{r}$$

It is customary to choose the potential to be zero at $r_p = \infty$.

On the path from A (r_A) to P ($r_p = \infty$), $\mathbf{E}_v$ is parallel to ds (d**r** = ds)

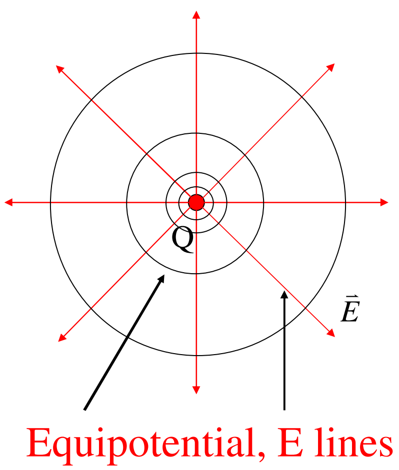
$$V_A = \int_A^P \vec{E}_v\cdot d\vec{s} = \int_A^P \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^2}\cdot\frac{\vec{r}}{r}\cdot d\vec{r} = \frac{Q}{4\pi\varepsilon_0}\cdot\int_A^P \frac{1}{r^2}\cdot\frac{\vec{r}}{r}\cdot d\vec{r} =$$

$$= \frac{Q}{4\pi\varepsilon_0}\cdot\int_{r_A}^{\infty}\frac{1}{r^2}\,dr = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_{r_A}^{\infty} = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{r_A}$$

V is constant on spherical surfaces centered on the point charge.

Electrostatics

Electric potential for a single point charges



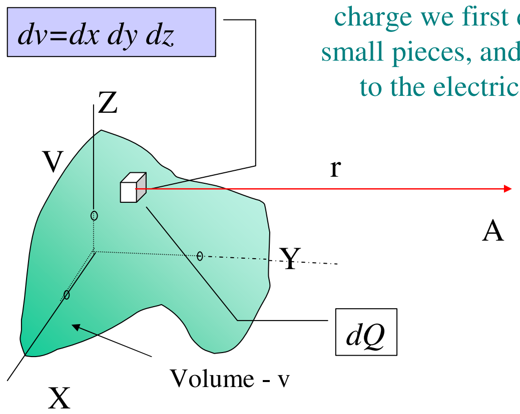
$$\vec{E}_v = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^2}\cdot\frac{\vec{r}}{r}$$

$$V = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{r}$$

Electrostatics

Potential produced by a continuous distribution of charge

In the case of a continuous distribution of charge we first divide the distribution up into small pieces, and then we sum the contribution, to the electric potential, from each piece:



$$dV_A = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dQ}{r} \cdot dv$$

$$V_A = \frac{1}{4\pi\varepsilon_0} \int_v \frac{dQ}{r}$$

Electrostatics

Potential produced by a continuous distribution of charges

Volume charges $dQ = \rho_v \cdot dv$

$$V_A = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_v}{r} dv \qquad \vec{E}_A = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_v}{r^2} \cdot \frac{\vec{r}}{r} dv$$

Surface charges $dQ = \rho_s \cdot dA$

$$V_A = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_s}{r} dA \qquad \vec{E}_A = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_s}{r^2} \cdot \frac{\vec{r}}{r} dA$$

Line charges $dQ = \rho_l \cdot dl$

$$V_A = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_l}{r} ds \qquad \vec{E}_A = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_l}{r^2} \cdot \frac{\vec{r}}{r} ds$$

Electrostatics

Field and electric potential

$$V_A(x,y,z)=\int_{A(x,y,z)}^{P(fix)} \vec{E_v}\cdot d\vec{s}$$

$$dV_A(x,y,z)=-\vec{E}_v\cdot d\vec{s}=-(E_{vx}\cdot\vec{i}+E_{vy}\cdot\vec{j}+E_{vz}\cdot\vec{k})\cdot(dx\cdot\vec{i}+dy\cdot\vec{j}+dz\cdot\vec{k})$$

$$dV_A(x,y,z)=-E_{vx}\cdot dx-E_{vy}\cdot dy-E_{vz}\cdot dz$$

But:

$$dV_A(x,y,z)=-\frac{\partial V}{\partial x}\cdot dx-\frac{\partial V}{\partial y}\cdot dy-\frac{\partial V}{\partial z}\cdot dz$$

$$E_{vx}=-\frac{\partial V}{\partial x};\quad E_{vy}=-\frac{\partial V}{\partial y};\quad E_{vz}=-\frac{\partial V}{\partial z}$$

Electrostatics

Field and electric potential

$$E_{vx}=-\frac{\partial V}{\partial x};\quad E_{vy}=-\frac{\partial V}{\partial y};\quad E_{vz}=-\frac{\partial V}{\partial z}$$

$$\vec{E_v}=-\frac{\partial V}{\partial x}\cdot\vec{i}-\frac{\partial V}{\partial y}\cdot\vec{j}-\frac{\partial V}{\partial z}\cdot\vec{k}=-\nabla V=-gradV$$

Important !!

Electrostatics

Laplace and Poisson equations

$$\int_{\Sigma} \vec{E}_v \cdot d\vec{A} = \frac{Q_{\Sigma}}{\varepsilon_0}$$

$$Q_{\Sigma} = \int_{V_{\Sigma}} \rho_v \cdot dv$$

$$\int_{\Sigma} \vec{E}_v \cdot d\vec{A} = \frac{1}{\varepsilon_0} \int_{V_{\Sigma}} \rho_v \cdot dv$$

$$\int_{\Sigma} \vec{E}_v \cdot d\vec{A} = \int_{V_{\Sigma}} div\vec{E} \cdot dv = \frac{1}{\varepsilon_0} \int_{V_{\Sigma}} \rho_v \cdot dv$$

Electrostatics

Laplace and Poisson equations

$$\int_{\Sigma} \vec{E}_v \cdot d\vec{A} = \int_{V_{\Sigma}} div\vec{E}_v \cdot dv = \frac{1}{\varepsilon_0} \int_{V_{\Sigma}} \rho_v \cdot dv$$

$$div\vec{E}_v = \frac{\rho_v}{\varepsilon_0}$$

$$\vec{E}_v = -gradV$$

$$div(-gradV) = \frac{\rho_v}{\varepsilon_0}$$

$$\nabla^2 V = \Delta V = -\frac{\rho_v}{\varepsilon_0}$$

Free space – Laplace equation

$$\rho_v = 0$$

Materials – Poisson equation

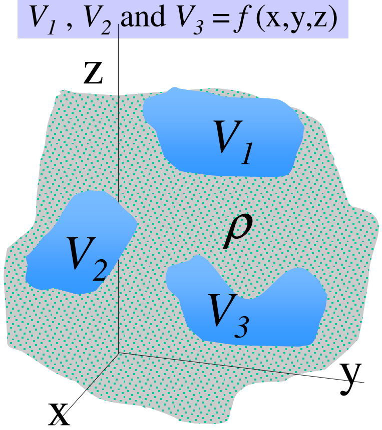
$$\rho_v \neq 0$$

Electrostatics

Poisson/Laplace in 3 D

Spatial charge density: $\rho_v = f(x,y,z)$

Boundary conditions:
V_1, V_2 and $V_3 = f(x,y,z)$



Potential $V = f(x,y,z)$?

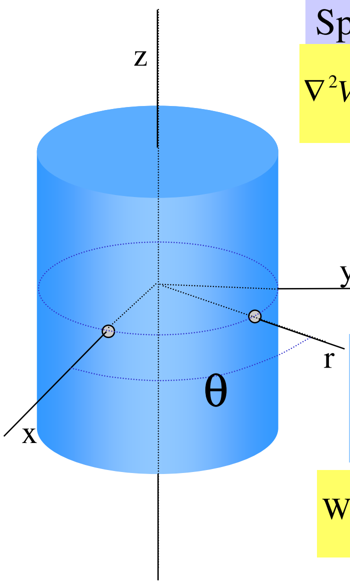
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right] V(x,y,z) = -\frac{\rho_v}{\varepsilon_0}$$

Solution of Laplace/Poisson will depend on boundaries.

Special cases:
- cylindrical geometry
- spherical geometry

Electrostatics

Poisson/Laplace in 3 D



Special case 1: Cylindrical geometry

$$\nabla^2 V = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right] V = -\frac{\rho_v}{\varepsilon_0}$$

If r – dependence only:
ρ_v and boundaries will be $f(r)$.

Thus: V will be $f(r)$ only

Example: $V=V_1$ at r_1 and V_2 at r_2, and $\rho_v = 0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)=0 \Rightarrow r\frac{dV}{dr}=c \Rightarrow \frac{dV}{dr}=\frac{c}{r}$$

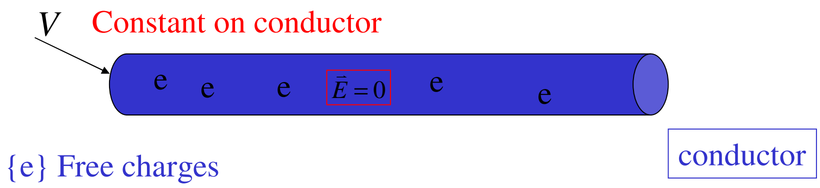
$$V(r) = c.\ln r + c' \qquad \text{(c and c': const.)}$$

With boundaries: $V(r) = V_1 + (V_2 - V_1)\dfrac{\ln r - \ln r_1}{\ln r_2 - \ln r_1}$

Electrostatics

Conductors in electrostatic field

- a conductor can **conduct**, or convey, electric charge.
- in static situations a conductor is a medium in which the **internal electric field is always zero**.
- it follows that **all parts of the conductor are at the same potential**.

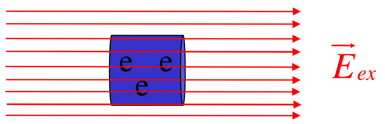


Electrostatics

Conductors in electrostatic field

Consider the conductor placed in an external electric field $\vec{E}_{ex}$

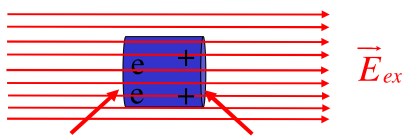
Time t = 0



Force on electrons $\vec{F} = -e \cdot \vec{E}_{ex}$

A very short time after $\vec{E}_{ex}$ is applied charge redistribution stops.

Time t > 0



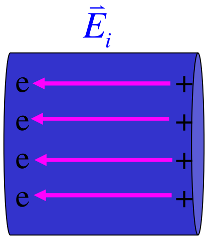
Build up of negative charges

Build up of positive charges

Electrostatics

Conductors in electrostatic field

The separation of charge produces an internal electric field $\vec{E}_i$ to the conductor



Redistribution of charge continues until the internal electric field magnitude equals that of the applied electric field.

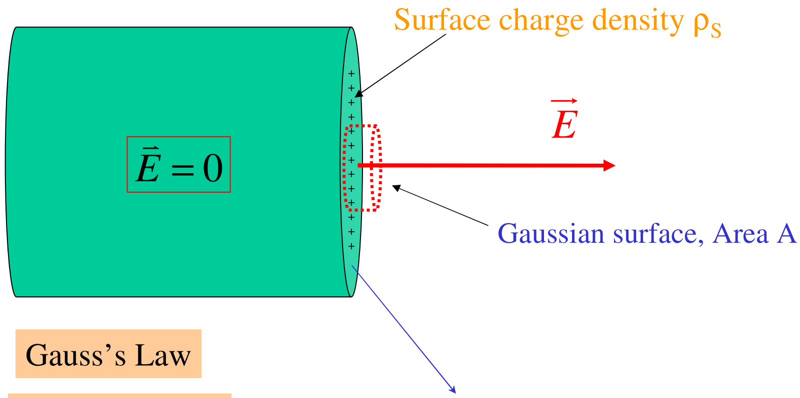
$$\vec{E}_{net} = \vec{E}_{ex} + \vec{E}_i$$
$$\vec{E}_{net} = 0$$

Inside conductor the net electric field is zero after charge redistribution

Time t ≈ 10^{-10} s

Electrostatics

Conductors in electrostatic field



Gauss's Law

$$E \cdot A = \frac{\rho_S A}{\varepsilon_o}$$

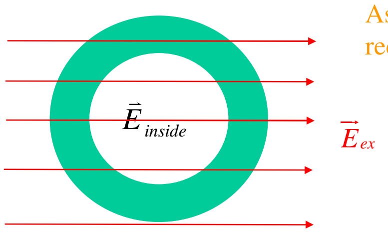
$$\rho_S = E \cdot \varepsilon_0$$

If E = constant, ρ_S constant !!

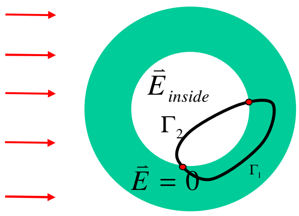
Electrostatics

Conductors in electrostatic field

Consider a spherical conducting shell. What is the electric field in the inside of the shell when an external electric field is applied to the shell and all charges have been redistributed?



Assume that $\vec{E}_{inside} \neq 0$ after charge redistribution has occurred.



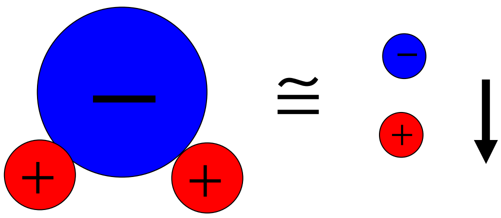
$$\oint_{\Gamma} \vec{E} \cdot d\vec{s} = \oint_{\Gamma_1} \vec{E} \cdot d\vec{s} + \oint_{\Gamma_2} \vec{E}_{inside} \cdot d\vec{s} = 0$$

0 since $\vec{E} = 0$

Only way to get 0 for all paths Γ_2 is to have $\vec{E}_{inside} = 0$

Electrostatics

The electric dipole



Dipole moment of selected molecules

$$H_2^{(+)}O^{(-)} = 6.1 \times 10^{-30}\,Cm$$
$$H^{(+)}Cl^{(-)} = 3.4 \times 10^{-30}\,Cm$$
$$N^{(-)}H_3^{(+)} = 5.0 \times 10^{-30}\,Cm$$

The dipole we know water

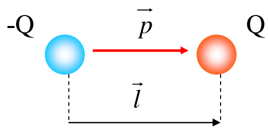
Electrostatics

The electric dipole

An electric dipole is formed by two equal but opposite polarity charges, which are in close proximity.

A dipole is characterized by its **electric dipole moment**:

$$\vec{p} = Q \cdot \vec{l} \quad , C \cdot m$$



We will consider the field of the dipole at distances much larger than its own length l, which is by definition infinitesimal.

Electrostatics

The electric dipole

Thus, the potential of an electric dipole is:

$$V_A \approx \frac{Ql\cos\theta}{4\pi\varepsilon_0 r^2} \approx \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$$

Note: the potential V drops off as $1/r^2$ for a dipole

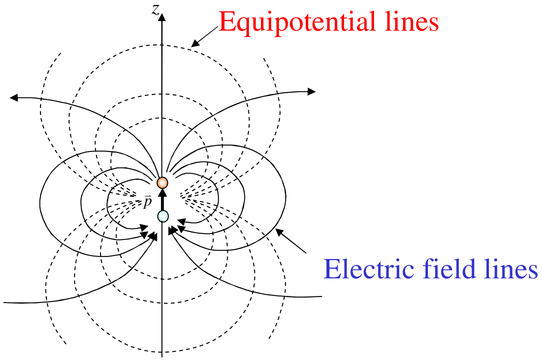
Using the expression for the gradient in spherical coordinates (due to the symmetry), the electric field vector is obtained:

$$\vec{E} = -gradV_A = -grad\left(\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}\right) = \frac{Ql}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\cdot\vec{u}_R + \sin\theta\cdot\vec{u}_\theta\right)$$

Note: the electric field of a dipole decrease with distance as $1/r^3$

Electrostatics

The electric dipole



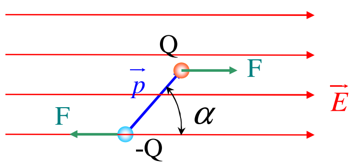
$$V_A = \frac{\vec{p} \cdot \vec{r}}{4\pi\varepsilon_0 r^3}$$

$$\vec{E} = \frac{Ql}{4\pi\varepsilon_0 r^3}\left(2\cos\theta \cdot \vec{u}_R + \sin\theta \cdot \vec{u}_\theta\right)$$

Electrostatics

The electric dipole

The dipole in an external electric field



The dipole tends to rotate so that its moment aligns with the external field.

The torque is:

$$\vec{m} = \vec{l} \times \vec{F} = \vec{l} \times \left(\vec{E} \cdot Q\right) = \left(\vec{l} \cdot Q\right) \times \vec{E} = \vec{p} \times \vec{E} \Rightarrow m = p \cdot E \cdot \sin\alpha, \quad Nm$$

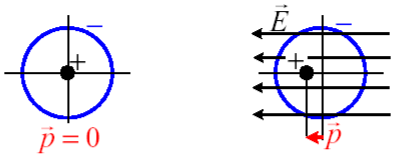
This, the electric dipole moment is the torque experienced by the dipole in an external field of unit intensity and direction orthogonal to the dipole length.

Electrostatics

Dielectric materials and polarization

Dielectrics have very low (negligible) DC conductivity. Their charges are not mobile, they are strongly bound to the atoms. Such charge is called bound charge, as opposed to the free charge in conductors.

External electric fields influence the dielectric atoms and molecules despite the fact that their charges are more or less fixed. Microscopic displacement of the centre of the electron cloud makes the atom look like a dipole.



Electrostatics

Dielectric materials and polarization

The above examples illustrates the so called electronic polarization, which occurs in dielectrics whose atoms and molecules are originally neutral. The process of polarization is balanced by the Coulomb attractive force.

Ionic polarization occurs in molecules consisting of positively and negatively charged ions, which are originally mixed and have zero net charge.

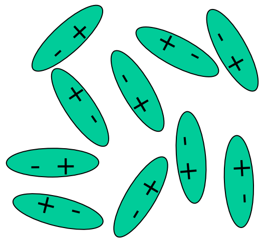
Orientational polarization occurs in materials, which consist of sub-domains with permanent microscopic separation of charges (electrets, polar liquids, etc.)

Each microscopic polarized region is characterized by its dipole moment:

$$\vec{p} = Q \cdot \vec{l}$$

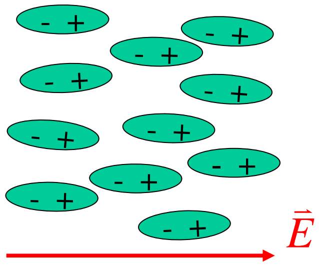
Electrostatics

Dielectric materials and polarization



Thermal agitation randomizes the molecules orientations. The average value of the dipole moments taken over any volume is zero: $\sum \vec{p}_i = 0$

No external electric field applied



An external electric field produces a torque on each of the dipole moments aligning them with the electric field lines.

$$\sum \vec{p}_i \neq 0$$

External electric field applied

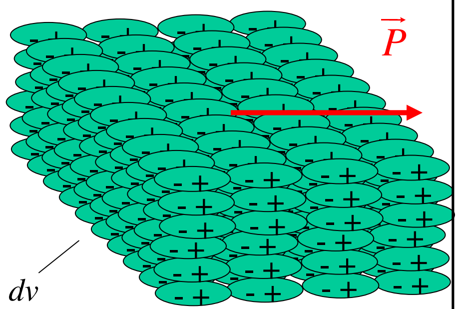
Electrostatics

Dielectric materials and polarization

To quantify the polarization effect on a macroscopic level, the polarization vector $\vec{P}$ as the dipole moment per unit volume is defined:

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_i \vec{p}_i}{\Delta v} = \frac{d\vec{p}}{dv} \quad ,C/m^2$$

$$d\vec{p} = \vec{P} \cdot dv$$

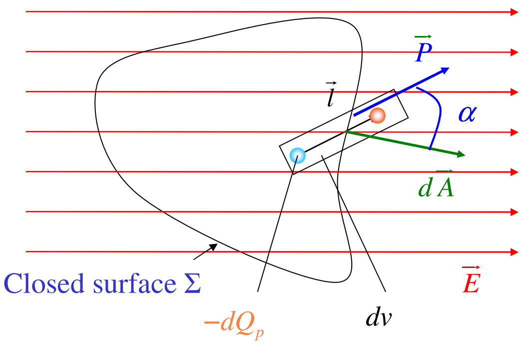


If a piece of dielectric is immersed in external electric field, the dipoles will align (more or less) with the field.

The internal charges will compensate each other but uncompensated surface charges will appear at both sides of the dielectric.

Electrostatics

Dielectric materials and polarization



$$-dQ_p = \frac{P \cdot dv}{l} = \frac{P \cdot dA \cdot l \cdot \cos\alpha}{l} = P \cdot \cos\alpha \cdot dA$$

$$-dQ_p = \vec{P} \cdot d\vec{A}$$

The polarization charges for the entry closed surface is:

$$Q_{p\Sigma} = -\oint\!\!\oint_{\Sigma} \vec{P} \cdot d\vec{A}$$

$$Q_{p\Sigma} = \iiint_{V_\Sigma} \rho_{vp} \cdot dv = -\oint\!\!\oint_{\Sigma} \vec{P} \cdot d\vec{A}$$

Equivalent polarization volume charge distributions

Electrostatics

Dielectric materials and polarization

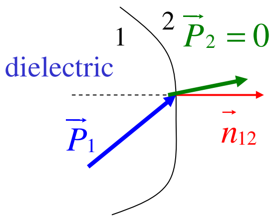
Thus, the polarization volume charge density is:

$$Q_{p\Sigma} = \iiint_{V_\Sigma} \rho_{vp} \cdot dv = -\iiint_{V_\Sigma} div\vec{P} \cdot dv \quad \Rightarrow \quad \rho_{vp} = -div\vec{P}$$

The polarization surface charge density will be: $\rho_{sp} = -div_S \vec{P}$

Where:

$$\rho_{sp} = -div_S \vec{P} = -\vec{n}_{12} \cdot \left(\vec{P}_2 - \vec{P}_1\right) = \vec{n}_{12} \cdot \vec{P}$$



$\vec{P}_2 = 0 \quad (in \quad vacumm)$

$\vec{n}_{12}$ is the unit vector to the interface

$\vec{P}_1 = \vec{P} \quad (in \;\; dielectric)$

Electrostatics

Electric field in dielectric materials

We now recognize two types of charges: free charges (in conductors) and bound charges (in dielectrics). The bound charges represents the behavior of the dielectrics atoms/molecule in vacuum.

We apply Gauss law for the electric flux density in vacuum in the presence of both types of charges:

$$\oiint_{\Sigma} \vec{E}\cdot d\vec{A} = \frac{Q_{f\Sigma}+Q_{p\Sigma}}{\varepsilon_0}$$

but:

$$Q_{p\Sigma} = -\oiint_{\Sigma} \vec{P}\cdot d\vec{A}$$

$$\oiint_{\Sigma} \vec{E}\cdot d\vec{A} = \frac{Q_{f\Sigma} - \oiint_{\Sigma} \vec{P}\cdot d\vec{A}}{\varepsilon_0} \Rightarrow \varepsilon_0 \cdot \left\{ \oiint_{\Sigma} \vec{E}\cdot d\vec{A} \right\} + \oiint_{\Sigma} \vec{P}\cdot d\vec{A} = Q_{f\Sigma}$$

Electrostatics

Electric field in dielectric materials

$$\varepsilon_0 \cdot \left\{ \oiint_{\Sigma} \vec{E}\cdot d\vec{A} \right\} + \oiint_{\Sigma} \vec{P}\cdot d\vec{A} = Q_{f\Sigma} \Rightarrow \oiint_{\Sigma} \left(\varepsilon_0\cdot\vec{E}+\vec{P}\right)\cdot d\vec{A} = Q_{f\Sigma}$$

Now we can define a vector, which depends on the free source charges only, and has nothing to do with the properties of medium:

$$\vec{D} = \varepsilon_0\cdot\vec{E}+\vec{P} \qquad [D]_{SI} = 1C/1m^2 \qquad \Rightarrow \oiint_{\Sigma} \vec{D}\cdot d\vec{A} = Q_{f\Sigma}$$

called electric flux density or electric displacement.

In an arbitrary medium (other that the vacuum) the electrostatic field is completely defined using 2 vector fields:

$$\vec{D}, \vec{E}$$

Electrostatics

Laws of electrostatics

The electric flux law
(generalization of the Gauss law valid only in vacuum)

Definition:

The total flux of the electric flux vector through a closed surface is equal to the total real free charge enclosed by the surface.

$$\oint\!\!\oint_{\Sigma} \vec{D} \cdot d\vec{A} = Q_{f\Sigma}$$ Integral form of the law

$$\oint\!\!\oint_{\Sigma} \vec{D} \cdot d\vec{A} = \iiint_{V_\Sigma} div\vec{D} \cdot dv = Q_{f\Sigma} = \iiint_{V_\Sigma} \rho_{vf} \cdot dv$$

$$\rho_{vf} = div\vec{D}$$ Differential form of the law

Electrostatics

Laws of electrostatics

The polarization law

From experimental studies, it is found that the polarization vector (the temporary component) is strongly related to the electric vector field. For most common dielectrics, these two vectors are collinear and proportional for a wide range of values of E (linear materials).

$$\vec{P}_t = \chi_e \cdot \varepsilon_0 \cdot \vec{E}$$ Valid just for linear materials

where: χ_e is the electric susceptibility of the material.

In this case using the relation between $\vec{E}, \vec{D}$ and $\vec{P}_t$ we will have:

$$\vec{D} = \varepsilon_0 \cdot \vec{E} + \vec{P}_t = \varepsilon_0 \cdot \vec{E} + \varepsilon_0 \cdot \chi_e \cdot \vec{E} = \varepsilon_0 \cdot \underbrace{(1+\chi_e)}_{\varepsilon_r} \cdot \vec{E}$$

$$\vec{D} = \varepsilon \cdot \vec{E}$$

ε_r Relative permitivity

Electrostatics

Laws of electrostatics

The polarization law

The dielectric constant ε_r is not really a constant. It may depend on frequency or on the field intensity. It is also called **relative dielectric permittivity**.

When the dielectric permittivity depends on the electric field E, it is said that the **medium is nonlinear**, because all the field relations become nonlinear equations.

When the dielectric permittivity depends on the position in the volume of the dielectric body $\varepsilon(x, y, z)$ it is said that the problem is **inhomogeneous**, as opposed to the homogeneous case when the properties of the material are constant throughout the volume.
Moreover, the dielectric properties may depend on the direction of the applied field because of certain properties of crystal lattices, etc. This is called **anisotropy** of the dielectric material.

Electrostatics

Laws of electrostatics

The polarization law

Then, the relation between the electric flux density vector and the electric field vector is a tensor one:

$$\vec{D} = \overline{\overline{\varepsilon}} \cdot \vec{E} \quad \Rightarrow \quad \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Fortunately, it is often suffices to assume that the medium is **homogeneous**, **linear** and **isotropic**. This is the simplest possible case.

Final note on the physical meaning of the relative dielectric permittivity: it shows how many times the electric field force is decreased in the volume of the dielectric due to the cancellation effect of the polarized microscopic dipoles.

Note: In general the polarization vector consist in 2 components:
- a temporary component (P_t) and a permanent one (P_p)

Electrostatics

Laws of electrostatics

The relation between D, E and P vectors

The vector sum between polarizations (both components) and the electrical field intensity, multiplied with the permitivity of the vacuum, is equal, at any moment and point, with the electrical flux density:

$$\vec{D} = \varepsilon_0 \cdot \vec{E} + \vec{P}_t + \vec{P}_p$$

For materials without permanent polarization: $\vec{D} = \varepsilon_0 \cdot \vec{E} + \vec{P}_t$

For linear materials without permanent polarization: $\vec{D} = \varepsilon \cdot \vec{E}$

For materials with anisotropy and without permanent polarization:

$$\vec{D} = \overline{\overline{\varepsilon}} \cdot \vec{E}$$

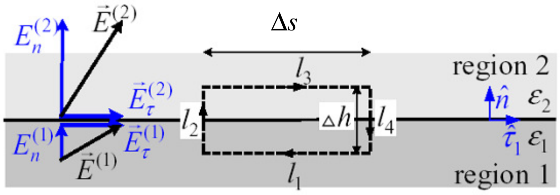
Electrostatics

Boundary conditions in electrostatics

Most practical problems involve more than two regions of different electric properties. The neighboring regions are separated by surface called boundaries (also interfaces). The field behavior at these boundaries is described by certain equations: **the boundary conditions.**

These equations are derived from the general field equations, which are valid in a volume. The boundary conditions are essential to the solution of any electrostatic field problem. These problems would have a unique solution only it its boundary conditions are specified.

The boundary conditions of the tangential to the interface field components



Electrostatics

Boundary conditions in electrostatics

$$\vec{n}\times\left[\vec{E}^{(2)}-\vec{E}^{(1)}\right]=0$$

The tangential to the interface components of the electric field vector are continuous across the material interface. !!

$$\vec{n}\cdot\left(D^{(2)}-D^{(1)}\right)=\rho_s$$

⇩

$$\varepsilon_2\cdot E_n^{(2)}-\varepsilon_1\cdot E_n^{(1)}=\rho_s$$!!!

At dielectric interface, if there is **not free surface charge**, then the *normal flux density will be continuous across the surface.*

Electrostatics

Boundary conditions in electrostatics

$$D_n^{(1)}=D_n^{(2)}$$

$$\varepsilon_1\cdot E_n^{(1)}=\varepsilon_2\cdot E_n^{(2)}$$

⇨

$$\frac{E_n^{(2)}}{E_n^{(1)}}=\frac{\varepsilon_1}{\varepsilon_2}$$

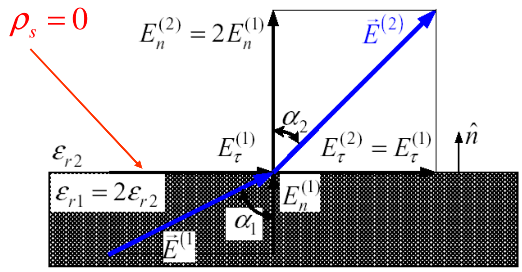
The normal electric field is discontinuous across dielectric interface.

Boundary conditions in electrostatics

At dielectric interface, the normal flux density and the tangential electric field are continuous. This would cause the electric field vector to change its direction abruptly (ONLY for surfaces without charge distributions):

$$E_{\tau_2}^{(1)} = E_{\tau_2}^{(2)} \quad \Rightarrow \quad E^{(1)} \cdot sin\,\alpha_1 = E^{(2)} \cdot sin\,\alpha_2$$

$$\varepsilon_1 \cdot E_n^{(1)} = \varepsilon_2 \cdot E_n^{(2)} \quad \Rightarrow \quad \varepsilon_1 \cdot E^{(1)} \cdot cos\,\alpha_1 = \varepsilon_2 \cdot E^{(2)} \cdot cos\,\alpha_2$$



$$\frac{tan\,\alpha_1}{tan\,\alpha_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

Very important !!

Capacitance

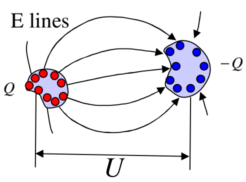
Capacitance is a property of a geometric configuration, usually two conducting objects separated by an insulating medium.

The system of two conductors, each carrying equal charge is known as a **capacitor.**

Capacitance is a measure of how much charge a particular configuration is able to retain when a battery of U volts is connected and then removed.

The amount of charge Q deposited on each conductor will be proportional to the voltage U of the battery and some constant C, called the capacitance.

By definition:



$$C = \frac{Q}{U}$$

Capacitance: Farad ={C/V}=F

Electrostatics

Capacitance

A general expression for the capacitance in terms of the E vector:

$$C = \frac{Q}{U} = \frac{\iint_{\Sigma} \vec{D} \cdot d\vec{A}}{\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}}$$

If the region surrounding the electrodes is homogeneous of dielectric permittivity, then the capacitance is expressed only in terms of the E vector:

$$C = \varepsilon \cdot \frac{\iint_{\Sigma} \vec{E} \cdot d\vec{A}}{\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}}$$

Electrostatics

Capacitance

If the capacitor is given, with a homogeneous dielectric, its capacitance is a function of the geometrical dimensions, which characterizes the shape, and the relative position of the plates and is direct proportional with the dielectric permitivity:

$$C = \varepsilon \cdot f\left(g_1, g_2, \ldots, g_n\right)$$

In the case of non-homogeneous dielectric media, the permitivity of the dielectric is different, and the capacitance is a function of the following form:

$$C = f\left(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n; g_1, g_2, \ldots, g_n\right)$$

Electrostatics

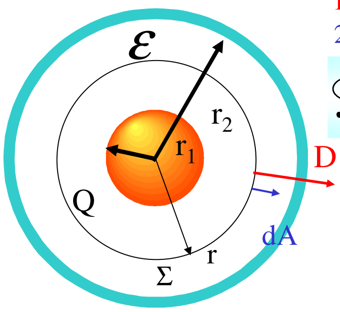
Calculation of the capacitance

Algorithm of the capacitance computation:

- Suppose the two conductors charged
- Compute the electric field intensity
- Compute the electric voltage between the two conductors
- Apply the capacitance definition

Electrostatics

Capacitance of spherical capacitor



1. Supposed the conductors charged with Q and -Q
2. Apply the flux law.

$$\oint\!\!\oint_{\Sigma} \vec{D}\cdot d\vec{A} = \iint_{\Sigma} D\cdot dA = D\cdot \iint_{\Sigma} dA = 4\cdot\pi\cdot r^2\cdot D = Q$$

$$D = \varepsilon\cdot E \qquad E = \frac{Q}{4\cdot\pi\cdot\varepsilon\cdot r^2}$$

3. Compute the voltage between the conductors

$$U_{AB} = \int_1^2 \vec{E}\cdot d\vec{s} = \int_1^2 E\cdot ds = \int_{r_1}^{r_2} \frac{Q}{4\cdot\pi\cdot\varepsilon\cdot r^2}\cdot dr = \frac{Q}{4\cdot\pi\cdot\varepsilon}\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{Q}{4\cdot\pi\cdot\varepsilon}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

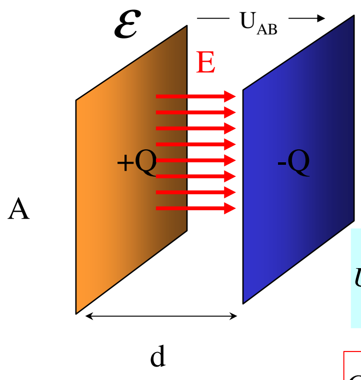
4. Apply the capacitance definition $C = \frac{Q}{U_{AB}}$

$$C = \frac{4\cdot\pi\cdot\varepsilon\cdot r_1\cdot r_2}{r_2 - r_1}$$

Electrostatics

If: $r_2 \rightarrow \infty$ $C = 4 \cdot \pi \cdot \varepsilon \cdot r_1$

Capacitance of parallel plates



The electric field inside the parallel plates:

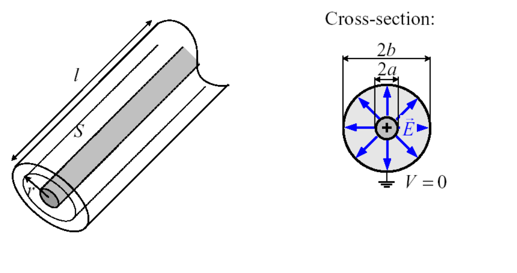
$$E = \frac{\rho_s}{\varepsilon} \qquad \rho_s = \frac{Q}{A} \quad \Rightarrow \quad E = \frac{Q}{A \cdot \varepsilon}$$

$$U_{AB} = \int_1^2 \vec{E} \cdot d\vec{s} = \int_1^2 E \cdot ds = E \cdot \int_1^2 ds = E \cdot d = \frac{Q \cdot d}{A \cdot \varepsilon}$$

$$C = \frac{Q}{U_{AB}} \qquad C = \frac{A \cdot \varepsilon}{d}$$

Electrostatics

Capacitance of cylindrical capacitor



Prove that:

$$C = \frac{2 \cdot \pi \cdot \varepsilon \cdot h}{ln \frac{b}{a}}$$

Electrostatics

Electrostatic field energy

In order to establish an electrical field in a space domain where this is initially null, it is necessary to move electrical charges from infinite to the bodies. The electrical field energy is equal to the total mechanical work needed to transport these charges.

In order to define the energy in such a way, some hypotheses have to be made:

the medium is isotropic, linear and without permanent polarization.

the storage of the charges on the conductors is made very slowly, in order to consider the field as being electrostatic and so that we don't have irreversible transformations of the mechanical work done in heat.

consider that the conductive system is immobile, such that we don't lose mechanical work to deform or move the conductors.

Electrostatics

Electrostatic field energy

Suppose *n* spherical conductors and the following additional assumptions:

All conductors are in the initial state without charges:

$$\begin{matrix} Q_i = 0 \\ V_i = 0 \end{matrix} \qquad \forall \quad i = 1, 2, .., n$$

The final state of the conductors will be:

$$\begin{matrix} Q_1, Q_2, ...Q_i, ...Q_n \\ V_1, V_2, ...V_i, ...V_n \end{matrix}$$

An intermediary state will be established proportionally, meaning that exist the following relations:

$$\begin{matrix} Q_i^{'} = \lambda \cdot Q_i \\ V_i^{'} = \lambda \cdot V_i \end{matrix} \qquad \forall \quad i = 1, 2, .., n$$

Electrostatics

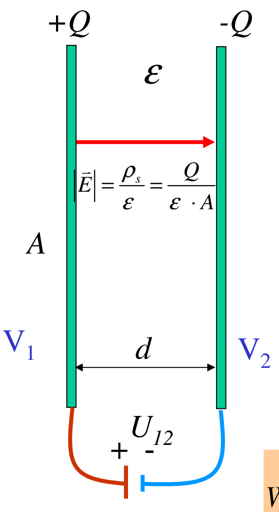
Electrostatic field energy

$$W_e = \frac{1}{2} \cdot \sum_{i=1}^{n} V_i \cdot Q_i$$

The above relation gives us the expression of the energy stored in the electrical field of some conductors that have the charges and potentials

Electrostatics

Electrostatic field energy



Consider a capacitor at potential difference U_{12} and of charges $+Q$, $-Q$ on the plates. Area of plates (A) and spacing (d).

$$W_e = \frac{1}{2} \cdot \sum_{i=1}^{n} V_i \cdot Q_i = \frac{1}{2} \cdot V_1 \cdot Q - \frac{1}{2} \cdot V_2 \cdot Q = \frac{1}{2} \cdot U_{12} \cdot Q$$

But: $C = \frac{Q}{U_{12}} = \frac{\varepsilon A}{d}$ $\quad W_e = \frac{1}{2} \cdot U_{12} \cdot Q = \frac{1}{2} \cdot C \cdot U^2{}_{12}$

$$U_{12} = E \cdot d$$

$$W_e = \frac{1}{2} \cdot \left(\frac{\varepsilon \cdot A}{d} \right) \cdot (E \cdot d)^2 = \frac{1}{2} \cdot \varepsilon \cdot A \cdot d \cdot E^2 = \frac{1}{2} \cdot \varepsilon \cdot V \cdot E^2$$

V is the volume between plates and NOT potential

Electrostatics

Electrostatic field energy

$$w_e = \frac{W_e}{V} = \frac{1}{2}\cdot\varepsilon\cdot E^2 = \frac{1}{2}\cdot\varepsilon\cdot E\cdot E = \frac{1}{2}\cdot D\cdot E$$

This relation is in opposite with the first one for the electrostatic field energy (which expresses the energy with respect to the potentials and charges and does not specify where it is located - on the conductors or inside the dielectric). *w_e is called as electrostatic energy density.*

In, general: $w_e = \frac{1}{2}\cdot\vec{D}\cdot\vec{E}$

The total electrical field energy is: $W_e = \iiint_V w_e \cdot dv = \frac{1}{2}\iiint_V \vec{D}\cdot\vec{E}\cdot dv$!!!!

Conclusion:

The electrical field energy is located inside the dielectric (wherever exists an electrical field) and not inside conductive bodies (where the field is zero).