The Theory of Electromagnetic Field

Chapter 2

Electrokinetic fields

Electrokinetic fields

Introduction

So far we have been discussing static electricity. The charges have not been moving. Now we want to see what happens when charges move. We are thus going to consider conductors, rather than insulators (in which charges cannot move). The idea of a charge in motion brings us immediately to the concept of electrical currents and magnetic fields. In this chapter we deal with some aspects of electrical currents.
This branch of electromagnetism is known as: electrokinetic fields.

Since we are considering moving charges we are no longer in electrostatic equilibrium, so the properties of conductors we say before no longer apply. In particular, when charges are moving the total electric field inside a conductor is no longer zero:

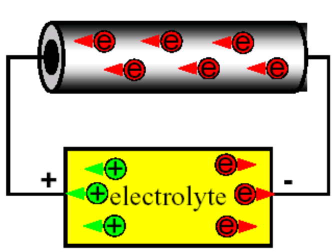
$$\vec{E} + \vec{E}_{external} \neq 0$$

Note: if a charge undergoes an acceleration it creates electromagnetic waves. This is the topic of the next chapters on electromagnetic fields.

Electrokinetic fields

Electric current

If the sources producing ESF are in physical contact with a metallic body, the electrons will immediately try to discharge them until there is no field inside the conductor. This would be exactly the case if the source is unable to provide continuously more and more charge through some mechanism for charge transfer.



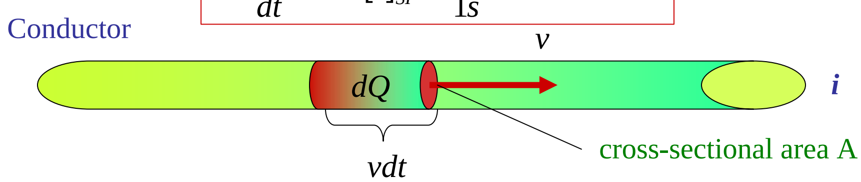
Electrokinetic fields

Electric current

If such a mechanism is provided, then a coordinated steady movement of charges occurs, which is called electric conductive current, or simply current *i (A)*.

The total amount of charges moving through a given cross section per unit time is the current, usually denoted by *i*:

$$i = \frac{dQ}{dt}, \qquad [i]_{SI} = \frac{1C}{1s} = 1A\,(Amper)$$



Electrokinetic fields

Electric current

If we consider the current per unit cross-sectional area, we get a value which can be defined in any point in space as a vector, typically denoted $\vec{J}$, called conductive current density:

$$\vec{J} = \lim_{\Delta A \to 0} \frac{\Delta i}{\Delta A} \cdot \vec{n} = \frac{di}{dA} \cdot \vec{n}, \qquad [J]_{SI} = \frac{1A}{1m^2}$$

where *n* is the normal direction of the plane.

The total current through the end face can be obtained from the current density as an integration over the cross-sectional area of the conducting medium.

$$i = \int_A \vec{J} \cdot d\vec{A}$$

Electrokinetic fields

Electric current

For the potential to rise along the direction of the current, there must be a source which converts some form of energy to electrical energy.
Examples of such sources are:

- Batteries
- Generators
- Thermocouples
- Photo-voltaic cells, etc.

In this case, inside the conductor the electrostatic equilibrium condition is not any more valid:

$$\vec{E} + \vec{E}_{emf} \neq 0$$

Where:

E is the electric field;

E_{emf} is the electric field established by the energy conversion.

Electrokinetic fields

Electric current

A steady state current flow requires a closed circuit. If we integrate along a closed circuit the above mentioned relation we will have:

$$e_{emf} = \oint_C \vec{E}_{emf} \cdot d\vec{s} \neq 0 \qquad \oint_C \vec{E} \cdot d\vec{s} = 0$$

Define the electromotive force (**emf**) or "voltage" of the battery as e_{emf}

Electric fields produces by stationary charges are conservative. Thus, they cannot by themselves maintain a steady current flow.

Note: until now we have discussed just about steady state conductive currents.

Additional with this current exist also a so called convection current.

Electrokinetic fields

Convection electrical current

The conduction electrical current has another property: it always passes through a conductive medium and the movement of the electrically charged particles is a relative movement with respect to the body.

If the electrical charge is transported directly by masses charges with electricity, an electrical current appears, in balance on these masses, called **convection electrical current**.

Consider a body – conductor or insulator – charged with an electrical charge of volume density ρ_v , which moves in a certain direction with the v velocity:

The convection current density is defined as: $\vec{J}_C = \rho_v \cdot \vec{v}$

and the corresponding total current is $i_c = \int_A \vec{J}_C \cdot d\vec{A}$

Electrokinetic fields

Ohm's law

From experimental studies, it is found that the conductive current density vector is strongly related to the electric vector field. For most common conductors, these two vectors are collinear and proportional for a wide range of values of E (linear materials).

The 1st local form: $\vec{J} = \sigma \cdot \vec{E}$ — Valid for linear materials without external electric field!!

Where σ is the conductivity of the conductors.

The 2nd local form: $\vec{E} = \frac{1}{\sigma} \cdot \vec{J} = \rho \cdot \vec{J}$ — Valid for linear materials without external electric field!!

Where ρ is the resistivity of the conductors. $\rho = \frac{1}{\sigma}$

Electrokinetic fields

Ohm's law

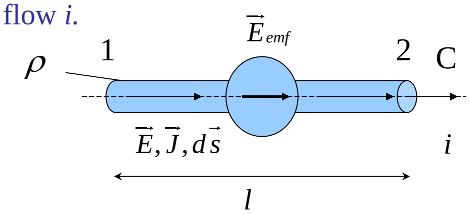
The 3rd local form: $\vec{E} + \vec{E}_{emf} = \rho \cdot \vec{J}$ Valid for linear materials with external electric field!!

The 4th local form: $\vec{J} = \sigma \cdot \left(\vec{E} + \vec{E}_{emf} \right)$ Valid for linear materials with external electric field!!

The integral form of Ohm's law.

Consider a piece of homogenous material of conductivity σ , length *l* and uniform cross section *A*, as shown below. Within the conductor *J*, *E* and E_{emf} are in the direction of current flow *i*.

$$\vec{E} + \vec{E}_{emf} = \rho \cdot \vec{J}$$



Electrokinetic fields

Ohm's law

The integral of the local form of Ohm's law along line *C* between the two end points (1) and (2) will be (note all the vectors are collinear):

$$\int_{(1)}^{(2)} \left(\vec{E} + \vec{E}_{emf} \right) \cdot d\vec{s} = \int_{(1)}^{(2)} \left(\rho \cdot \vec{J} \right) \cdot d\vec{s}$$

$$\int_{(1)}^{(2)} \left(\vec{E} + \vec{E}_{emf} \right) \cdot d\vec{s} = \int_{(1)}^{(2)} \vec{E} \cdot d\vec{s} + \int_{(1)}^{(2)} \vec{E}_{emf} \cdot d\vec{s} = u_{12} + e_{12}$$

u_{12} The electric voltage between the two end points 1 and 2

e_{12} The electromotive voltage between the two end points 1 and 2

Electrokinetic fields

Ohm's law

It is supposed that the current density is uniform over the cross section of the conductor

$$\int_{(1)}^{(2)} \left(\rho \cdot \vec{J}\right) \cdot d\vec{s} = \int_{(1)}^{(2)} \left(\rho \cdot \frac{i}{A}\right) \cdot ds = i \cdot \int_{(1)}^{(2)} \frac{\rho \cdot ds}{A}$$

R_{12} is the resistance of the conductor between (1) and (2)

$$R_{12} = \int_{(1)}^{(2)} \frac{\rho \cdot ds}{A}$$

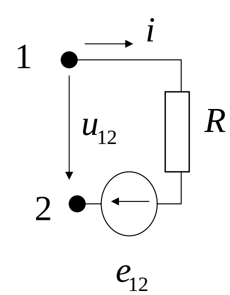
If, the cross section is constant along the whole line then the resistance between the two points (1) and (2) will be:

$$R_{12} = \int_{(1)}^{(2)} \frac{\rho \cdot ds}{A} = \frac{\rho}{A} \cdot \int_{(1)}^{(2)} ds = \frac{\rho \cdot l}{A}$$

Electrokinetic fields

The integral form of Ohm's law.

$$u_{12} + e_{12} = R_{12} \cdot i$$



$$u_{12} \pm e_{12} = i \cdot R$$

Electrokinetic fields

Charges conservation law (continuity law)

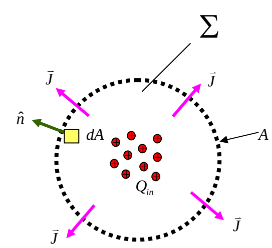
A fundamental law of physics is that charge can either be created nor destroyed. At the most, charges can be moved from one place to another by currents.

A <u>current outflow from a volume</u> inevitably means that <u>charges decreases in this volume</u>. A <u>current flow into a volume</u> implies that the enclosed amount of <u>charges increases.</u>

This is represented by the so-called continuity of current law (or charges conservation law). In integral form it is:

Electrokinetic fields

Law of conservation of charges



Integral form

$$i = -\frac{dQ}{dt} = \oiint_{\Sigma} \vec{J} \cdot d\vec{A}$$

$$\Rightarrow -\frac{d}{dt}\left(\oiiint_{V_\Sigma} \rho_v \cdot dv \right) = \oiint_{\Sigma} \vec{J} \cdot d\vec{A} = \oiiint_{V_\Sigma} div\vec{J} \cdot dv$$

<u>Very important remark</u>: when in the expression of an EM law in the **<u>integral form</u>** there is a derivative of an integral over a volume (or surface or line) there are two different possibilities:

The volume (or surface or line) is immobile.

The volume, surface or line are mobile with the velocity v = cst.

Electrokinetic fields

Law of conservation of charges

Suppose that the volume V is mobile with the velocity v, then the derivative with respect the time of the volume integral will be:

$$\frac{d}{dt}\left(\oiiint_{V_\Sigma} \rho_v \cdot dv\right) = \underbrace{\oiiint_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \cdot dv}_{\text{The volume is immobile}} + \underbrace{\oiint_{\Sigma} \rho_v \cdot \vec{v} \cdot d\vec{A}}_{\text{The volume is mobile}}$$

In this case the local form of the continuity law will be:

$$\oiint_{\Sigma} \vec{J} \cdot d\vec{A} = \oiiint_{V_\Sigma} div\vec{J} \cdot dv = -\frac{d}{dt}\left(\oiiint_{V_\Sigma} \rho_v \cdot dv\right) = -\oiiint_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \cdot dv - \oiint_{\Sigma} \rho_v \cdot \vec{v} \cdot d\vec{A}$$

$$\oiint_{\Sigma} \vec{J} \cdot d\vec{A} + \oiint_{\Sigma} \rho_v \cdot \vec{v} \cdot d\vec{A} = -\oiiint_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \cdot dv$$

Electrokinetic fields

Law of conservation of charges

$$\oiint_{\Sigma} \vec{J} \cdot d\vec{A} + \oiint_{\Sigma} \rho_v \cdot \vec{v} \cdot d\vec{A} = \oiint_{\Sigma} \left(\vec{J} + \rho_v \cdot \vec{v}\right) \cdot d\vec{A} = -\oiiint_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \cdot dv$$

$$\vec{J}_c = \rho_v \cdot \vec{v}, \quad (convection \quad current \quad density)$$

$$\vec{J}, \quad (conduction \quad current \quad density)$$

$$\oiint_{\Sigma} \left(\vec{J} + \vec{J}_c\right) \cdot d\vec{A} = -\oiiint_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \cdot dv$$

$$\oiint_{\Sigma} \left(\vec{J} + \vec{J}_c\right) \cdot d\vec{A} = \iiint_{V_\Sigma} div\left(\vec{J} + \vec{J}_c\right) \cdot dv = -\oiiint_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \cdot dv$$

$$div\left(\vec{J} + \vec{J}_c\right) = -\frac{\partial \rho_v}{\partial t}$$

General local form of the continuity law

Electrokinetic fields

Law of conservation of charges

$$div\left(\vec{J}+\vec{J}_c\right)=-\frac{\partial \rho_v}{\partial t}$$

This is the local form of the law of conservation of charges. It shows that the sources of convection and conduction current densities are points, at which the amount of charge density are changed with respect with time.

1) Particular case. The volume is immobile

$$\vec{J}_c=0$$

$$div\vec{J}=-\frac{\partial \rho_v}{\partial t}$$

This is the continuity law in differential form for immobile structures.

Electrokinetic fields

2) Particular case. The volume is immobile and we are in steady case

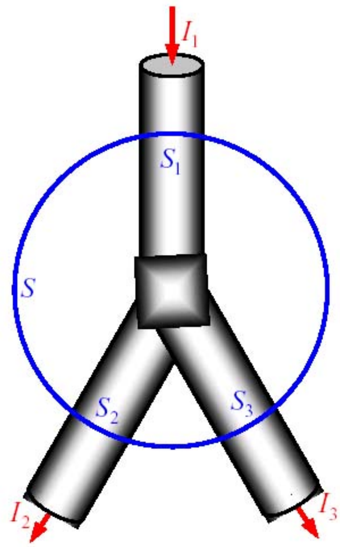
Here, we consider steady currents. In this case, the charge moves steadily (in average) and its density at a given point does not change in time. Therefore,

$$\frac{\partial \rho_v}{\partial t}=0 \Rightarrow div\vec{J}=0$$

$$\iiint_{V_\Sigma} div\vec{J}\cdot dv=\oiint_{\Sigma}\vec{J}\cdot d\vec{A}=0$$

Electrokinetic fields

Assume now the the integration is carried out over a volume of a node of wires. The currents are confined within the wires, so the surface integral is actually the sum of the integrals over the cross-sections of the individual wires.



$$\oiint_{\Sigma} \vec{J}\cdot d\vec{A} = \iint_{S_1} \vec{J}\cdot d\vec{A} + \iint_{S_2} \vec{J}\cdot d\vec{A} + \iint_{S_3} \vec{J}\cdot d\vec{A} = 0$$

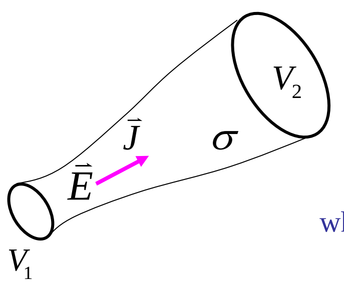
$$\Rightarrow I_1 + I_2 + I_3 = 0$$

This is exactly Kirchhoff's current law in circuit theory whose general form is:

$$\Rightarrow \sum_n I_n = 0$$

Electrokinetic fields

Resistance Here we present the general expression for obtaining the electrical resistance of an object.



$$R_{12} = \frac{u_{12}}{i} = \frac{V_1 - V_2}{i}$$

where $u_{12} = \int_1^2 \vec{E}\cdot\overrightarrow{ds}$

and

$$i = \int_{\Sigma} \vec{J}\cdot\overrightarrow{dA} = \int_{\Sigma} \sigma\cdot\vec{E}\cdot\overrightarrow{dA}$$

Electrokinetic fields

Resistance

A general expression for the resistance in terms of the E vector:

$$R_{12} = \frac{u_{12}}{i} = \frac{\int_1^2 \vec{E} \cdot d\vec{s}}{\iint_\Sigma \vec{J} \cdot d\vec{A}} = \frac{1}{\sigma} \cdot \frac{\int_1^2 \vec{E} \cdot d\vec{s}}{\iint_\Sigma \vec{E} \cdot d\vec{A}}, \quad 1\Omega = \frac{1V}{1A}$$

Be aware that the point (1) must be a point on the electrode of higher potential, and (2) is on the electrode of lower potential:

$$G_{12} = \frac{i}{u_{12}} = \sigma \cdot \frac{\iint_\Sigma \vec{E} \cdot d\vec{A}}{\int_1^2 \vec{E} \cdot d\vec{s}}, \quad 1S = \frac{1A}{1V} = 1\Omega^{-1}$$

Electrokinetic fields

Resistance

The analogy between the conductance and the capacitance is obvious:

$$C = \frac{Q}{U} = \varepsilon \cdot \frac{\iint_\Sigma \vec{E} \cdot d\vec{A}}{\int_1^2 \vec{E} \cdot d\vec{s}} \qquad G = \frac{i}{u} = \sigma \frac{\iint_\Sigma \vec{E} \cdot d\vec{A}}{\int_1^2 \vec{E} \cdot d\vec{s}}$$

Assume that there are two structures with exactly the same shape of the electrodes. The difference is that the region separating the electrodes is a dielectric in the first case, and a conductor in the second case. The ration of the capacitance and conductance is:

$$\frac{C}{G} = \frac{\varepsilon}{\sigma}$$

Resistance

The above formula is very convenient to find the resistance (or the conductance) of a structure, for which we already calculated the capacitance.

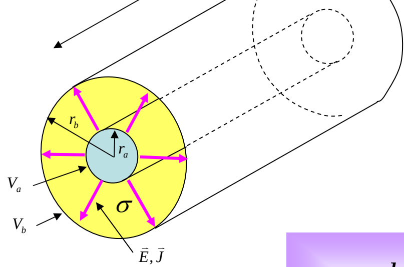
Example: the conductance of a parallel-plate structure is:

$$C = \frac{\varepsilon \cdot A}{d}, \quad F \qquad G = \frac{\sigma \cdot A}{d}, \quad \Omega^{-1} \qquad R = \frac{d}{A \cdot \sigma}, \quad \Omega$$

Where, A is the area of the plate (or the cross-section of the resistive rod) and d is the separation distance (or the length of the resistive rod)

Resistance

The expression for the resistance between the inner and outer cylindrical shells shown in the figure is:



$$C = \frac{2 \cdot \pi \cdot \varepsilon \cdot L}{ln \frac{r_b}{r_a}}$$

$$G = \frac{2 \cdot \pi \cdot \sigma \cdot L}{ln \frac{r_b}{r_a}}$$

$$R = \frac{\rho \cdot ln \frac{r_b}{r_a}}{2 \cdot \pi \cdot L}$$

Electrokinetic fields

Analogy between the electrostatic field and electrokinetic field

$$U_{AB} = \int_A^B \vec{E} \cdot d\vec{s}$$ $$u_{AB} = \int_A^B \vec{E} \cdot d\vec{s}$$

$$div\vec{D} = \rho_v$$ $$\oiint_\Sigma \vec{D} \cdot d\vec{A} = Q_{f\Sigma}$$ $$\oiint_\Sigma \vec{J} \cdot d\vec{A} = i$$ $$div\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

$$div_S \vec{D} = \vec{n} \cdot \left(\vec{D}_2 - \vec{D}_1\right) = \rho_s$$ $$div_S \vec{J} = \vec{n} \cdot \left(\vec{J}_2 - \vec{J}_1\right) = -\frac{\partial \rho_s}{\partial t}$$

$$\vec{D} = \varepsilon \cdot \vec{E}$$ $$\vec{J} = \sigma \cdot \vec{E}$$

$$\vec{D} = \varepsilon \cdot \vec{E} + \vec{P}_p$$ $$\vec{J} = \sigma \cdot \vec{E} + \sigma \cdot \vec{E}_{emf}$$

$$C = \frac{Q}{U}$$ $$G = \frac{i}{u}$$

Electrokinetic fields

Analogy between the electrostatic field and electrokinetic field

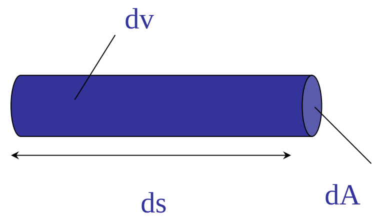
$$U_{AB} \Leftrightarrow u_{AB}$$
$$\vec{E} \Leftrightarrow \vec{E}$$
$$\vec{D} \Leftrightarrow \vec{J}$$
$$Q \Leftrightarrow i$$
$$\varepsilon \Leftrightarrow \sigma$$
$$\vec{P}_p \Leftrightarrow \sigma \cdot \vec{E}_{emf}$$
$$C \Leftrightarrow G$$

The analogy is very useful for many practical problems. The electrostatic problems are usually more easier to be solved in comparison with the equivalent electrokinetic problems.

Electrokinetic fields

Power density and Joule's law

Consider an infinitesimal volume *dv* of a resistive material



The work done by the electric field to move an infinitesimal charge *dQ* from one end to the other is:

$$d^2W = dV \cdot dQ = \left(\vec{E} \cdot d\vec{s}\right) \cdot dQ, \quad Joule$$

The power is defined as the rate of changes of energy with time. The power required for this charge transfer is:

$$dP = \frac{d^2W}{dt} = \frac{\left(\vec{E} \cdot d\vec{s}\right) \cdot dQ}{dt} = \left(\vec{E} \cdot d\vec{s}\right) \cdot i = \left(\vec{E} \cdot d\vec{s}\right) \cdot \left(\vec{J} \cdot d\vec{A}\right)$$

Electrokinetic fields

Power density and Joule's law

The power needed to move the charges in the infinitesimal volume *dv* can be also written as:

$$dP = \left(\vec{E} \cdot d\vec{s}\right) \cdot \left(\vec{J} \cdot d\vec{A}\right) = \left(\vec{E} \cdot \vec{J}\right) \cdot d\vec{s} \cdot d\vec{A} = \left(\vec{E} \cdot \vec{J}\right) \cdot dv$$

The power density is defined as the power per unit volume spent by the electric field in moving the charges across it:

$$p = \frac{dP}{dv} = \vec{E} \cdot \vec{J} = \sigma \cdot J^2 = \rho \cdot E^2 \quad W/m^3$$

Joule's law states that for a given volume V_Σ the total electric power used to move charges in the whole conductor, converted into heat is:

$$P = \iiint_{V_\Sigma} p \cdot dv = \iiint_{V_\Sigma} \left(\vec{E} \cdot \vec{J}\right) \cdot dv, \quad W$$

Electrokinetic fields

Power density and Joule’s law

In a conductor of uniform cross section $dv = dAds$, with ds measured in the direction of J. The above equation becomes:

$$P = \iiint_{V_\Sigma} \left(\vec{E}\cdot\vec{J}\right)\cdot\left(d\vec{A}\cdot d\vec{s}\right) =$$

$$= \iiint_{V_\Sigma} \left(\vec{E}\cdot d\vec{s}\right)\cdot\left(\vec{J}\cdot d\vec{A}\right) = \underbrace{\left(\int_s \vec{E}\cdot d\vec{s}\right)}_{u_{12}}\underbrace{\left(\iint_A \vec{J}\cdot d\vec{A}\right)}_{i}$$

Where, i is the current in the conductor.

$$P = u_{12}\cdot i$$

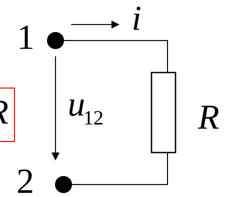
This is the integral form the Joule’s (Lenz) law for a circuit branch without source:

Electrokinetic fields

Power density and Joule’s law

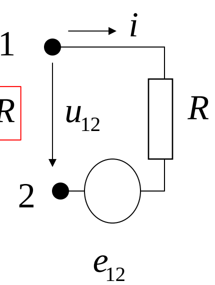
The expression of power dissipation is:

$$u_{12} = i\cdot R$$



$$P = u_{12}\cdot i \qquad P = R\cdot i^2$$

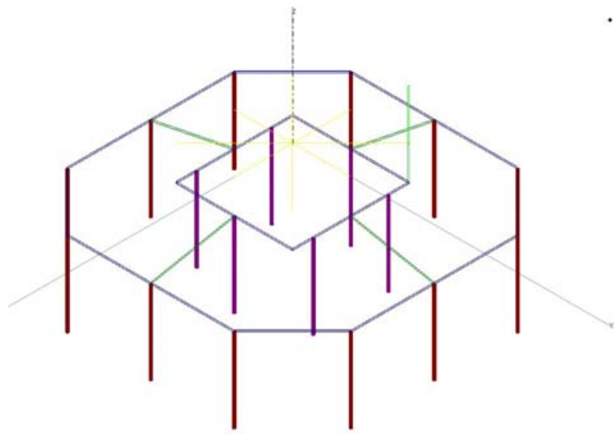
$$u_{12} \pm e_{12} = i\cdot R$$



$$P = \left(i\cdot R \mp e_{12}\right)\cdot i = i^2\cdot R \mp e_{12}\cdot i$$

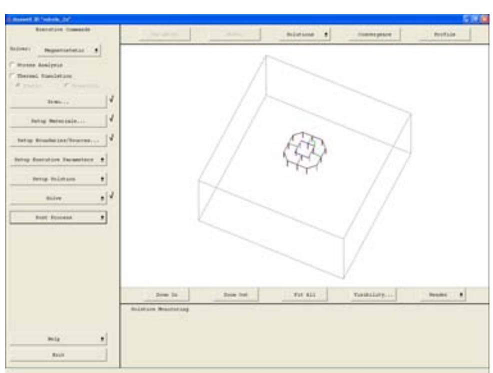
Electrokinetic fields

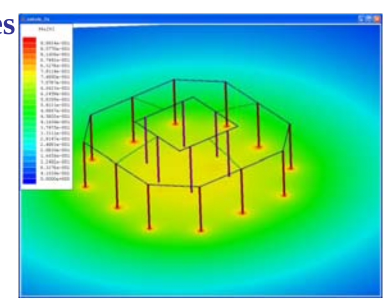
Examples

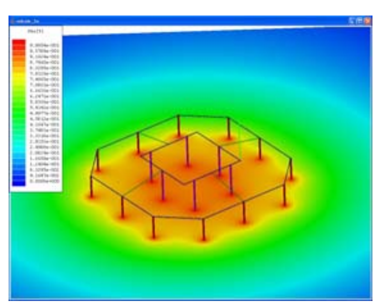


Electrokinetic fields

Examples







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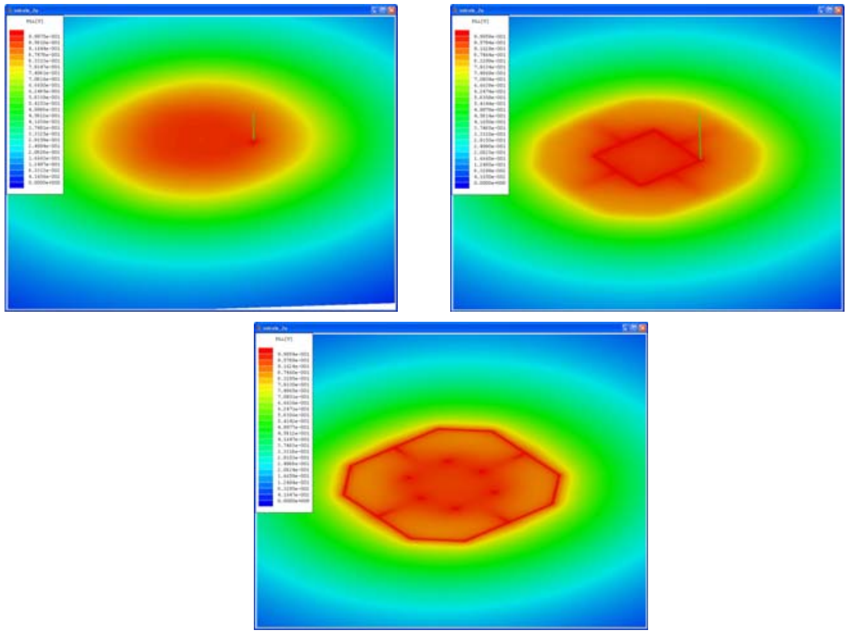

Electrokinetic fields